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MATHEMATICIANS AND THE NEW SELECTIVE SERVICE REGULATIONS

On January 10, 1944, Selective Service Headquarters released amended Local Board Memoranda 115 and 115B, which become effective on February 1, and also an amended Activity and Occupational Bulletin No. 33-6, which becomes effective on February 15. These vitally concern all chairmen of departments of mathematics for they deal with the occupational classification of both teachers and undergraduate students. This communication contains a summary of the provisions of these directives.

Mathematicians are still included in the List of Critical Occupations. It is specifically stated that the titles included in Part II (Professional and Scientific Occupations) of the List of Critical Occupations "shall be considered as also including persons engaged in full-time teaching of these professions. A person may be considered as engaged in full-time teaching if he devotes not less than 15 hours per week in contact with students in actual classroom or laboratory instruction." The problem of securing the proper classification of a *teacher* of mathematics differs in accordance with the age group into which he falls.

I. *Deferment of Mathematicians in 18–21 Age Group*

All requests for new or additional occupational deferments for registrants in this age group who are teachers of mathematics must be made on Director of Selective Service Form 42-A Special. An original and two copies of this form must be presented by the employer to the State Director in whose state is located the registrant's principal place of employment.

Effective February 1, no registrant in this group, at the time he is classified, may be considered as a "necessary man" entitled to be placed in Class II-A or Class II-B unless:

(a) the State Director of Selective Service has endorsed his Form 42-A Special with a statement that, based on the information furnished therein, he recommends that the local board except the registrant from the general restrictions against occupational deferment of registrants aged 18 through 21.

(b) he is classified as (1) belonging to the Personnel of the Merchant Marine or Army Transportation Corps or (2) as a student who qualifies under AOB 33-6.

Registrants in this group who are classified II-A or II-B on February 1, 1944, will, in general, not have deferments terminated before the expiration date. (This, however, does not apply to students covered in the new AOB 33-6.)

The President and Secretary of the Society have addressed a memorandum to State Directors of Selective Service, calling attention to the vital need for these mathematicians to handle the mathematics in the Army Specialized Training Program and the Navy College Training Program.

II. *Deferment of Mathematicians in 22–37 Age Group*

In making requests for occupational deferment, one copy of Form 42-A must be filed with the local board, as heretofore. The local board may, without going further, classify the registrant in II-A or II-B, as requested.

If the registrant is not classified in II-A or II-B and if reference to the United States Employment Service was not made by the local board prior to classification in I-A, I-A-O or IV-E, it must be made immediately after such classification. Upon suitable certification from the USES, the local board must reopen the case. The cases of registrants who are qualified for professional and scientific occupations will be forwarded by the USES to the National Roster. In all cases in which reference to the USES is made, the local board is directed not to issue an order to report for induction until it has received a report from the USES or until the expiration of 30 days after referral, whichever occurs first. If, during the 30-day period, the USES certifies to the local board that the registrant is qualified and that his removal would adversely affect the maintenance of his employer's required production, the local board must reopen the case and consider the new evidence as a basis for further deferment.

Even though a registrant is not employed in the area in which his local board is located, the registrant's local board must refer the case to the local USES office in the local board area and any further reference of the case to the USES office in which the registrant is employed will be undertaken and accomplished by the USES.

In view of these provisions, *department chairmen are urged to register their needs for personnel with the USES office where the college or university is located and in the USES office where the registrant's local board is located.* It is probable that the USES will be unable to supply any mathematicians but this is the best way to establish the fact that there is a shortage. If the USES does not have calls for mathematicians, physicists, engineers, chemists, et cetera, it will have no reason to believe that there is a shortage.

The USES offices are required to refer cases involving scientific personnel to the National Roster. (Information has come to the Secretary to the effect that the USES offices are not now referring cases to the Roster.) If department chairmen believe that the advice given by the local USES office to the local board is inadequate or incorrect, the employer is advised to call the attention of the USES office to the instructions in Sections 5300–5309 of the USES Manual (which contain instructions involving the National Roster). If Roster advice is not sought, the chairman will do well to report all facts in the case directly to the Roster.

Department chairmen are urged to make the original requests for occupational deferment as strong as possible and to make all possible appeals. Despite the curtailment of certain college training programs, chairmen are advised to continue requests for deferment, for the shortage of qualified mathematicians is still critical.

III. *Department of Undergraduates*

In Activity and Occupation Bulletin 33-6 the position is taken that the Army and Navy Training Programs will provide adequately for the needs of the armed forces. Student deferment is for civilian needs in war production and in support of the war effort.

Students who will graduate on or before July 1, 1944, in a considerable number of fields (including engineering branches, bacteriology, chemistry, forestry, geophysics, mathematics, meteorology, optometry, pharmacy, and physics including astronomy) are eligible for occupational deferment on certification by the institution and the National Roster.

An undergraduate who is a full-time student of chemistry, engineering, geology, geophysics, or physics who will graduate after July 1, 1944, will be eligible for occupational deferment, if he is properly certified by his institution and the National Roster and will graduate within 24 months from the date of certification and provided the national quota of 10,000 is not exceeded. Mathematics was not included in the list mentioned above, in spite of the strong protests of our Washington representatives. For undergraduate students in the 18–21 age group, requests for deferment are made in duplicate on DSS Form 42 Special; for students over 22 years of age, Form 42 is used.

A national quota of 10,000 has been established for students who should be occupationally deferred at any one time by reason of pursuing courses of study in chemistry, engineering, geology, geophysics, and physics. Students deferred to graduate on or before July 1, 1944, and students deferred for reasons other than pursuing a course of study will not be counted against this quota.

Requests for information needed by the National Roster to set up institutional quotas, et cetera, have already gone out to college presidents. Information regarding the procedures to be followed by institutions and departments concerned will be issued by the National Roster to college presidents in the near future. The departments concerned should be on the lookout for such information.

* * * * * * * *

Department chairmen are requested to inform the Secretary of cases in which mathematicians are improperly classified. In this manner the War Policy Committee, through its Washington representatives, will be better able to present the needs of the mathematicians.

M. H. STONE,
President
J. R. KLINE,
Secretary

January 17, 1944.

MAPPINGS BY MEANS OF SYSTEMS OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

W. T. MARTIN

1. **Introduction.** An analytic mapping of a domain D in the space E_{2k} of k complex variables $z_1, \cdots, z_k$ is a mapping defined by

$$T: \quad z_j' = f_j(z_1, \cdots, z_k), \qquad j = 1, \cdots, k, \tag{1.1}$$

where the $f_j(z)$ are analytic in D. We shall consider only univalent (schlicht) domains D contained in the finite portion of the space E_{2k}. An important result in the theory of analytic mappings states that the mapping is topological (that is, 1-1 and bi-continuous) if, and only if, the Jacobian

$$J(z) \equiv \partial(f_1, \cdots, f_k)/\partial(z_1, \cdots, z_k) \tag{1.2}$$

is different from zero at each point of D (Carathéodory [12]).[1]

An analytic mapping T is called an *inner mapping* of a domain D if $TD \subset D$. It is called an *automorphism* of D if T is 1-1 and if $TD = D$.

In 1907 Poincaré [17] showed that, given two domains D and D', it is not always possible to map D onto D' analytically. Since that time, several general problems have been considered. One of these problems is to indicate some general rules which tell whether or not two given domains can be mapped analytically upon each other. A second problem is to determine a family of special domains, in terms of some simple properties, the family to be such that every domain can be mapped analytically onto one of these special domains.

In this talk I shall deal with a special case of the first of these two problems together with certain results on the second problem. The work on the first problem which will be presented is based upon Henri Cartan's theory of mappings of domains onto domains of circular type. The work given on the second problem is based upon Bergman's theory of representative domains. In preparing this talk, I have used freely the excellent résumé on analytic mapping contained in the book on several complex variables by Behnke and Thullen [4]. I have also used freely material from the manuscript by Bochner and the speaker of a book on several complex variables, now in preparation [10]. I am indebted to Professor Bochner for permission to use

An address delivered before the New York meeting of the Society on October 30, 1943, by invitation of the Program Committee; received by the editors November 9, 1943.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

material from this manuscript. The approach used in much of this talk follows the approach of that manuscript, and in addition certain so far unpublished results from it are given here.

Let D be a domain containing the origin, and let (1.1) be an analytic mapping of D onto a domain D', with a fixed origin. We shall frequently write (1.1) in the form

$$T: \quad z_j' = a_{1j}z_1 + \cdots + a_{kj}z_k + \text{(higher powers)}, \qquad j = 1, \cdots, k, \tag{1.3}$$

to mean that the analytic functions $f_j(z)$ defining T have the expansions

$$f_j(z) = a_{1j}z_1 + \cdots + a_{kj}z_k + \text{(higher powers)} \tag{1.4}$$

in a neighborhood of the origin.

Again, if D is a domain containing the origin and if $f(z_1, \cdots, z_k)$ is analytic in D we shall sometimes have occasion to develop f in the neighborhood of the origin in terms of homogeneous polynomials (a diagonal series)

$$f(z_1, \cdots, z_k) = \sum_{n=0}^{\infty} P_n(z_1, \cdots, z_k). \tag{1.5}$$

The expression $P_n(z)$ is a homogeneous polynomial of degree n and is given by

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z_1e^{i\theta}, \cdots, z_ke^{i\theta})e^{-in\theta}d\theta \tag{1.6}$$

for z in a sufficiently small neighborhood N of the origin. From (1.6) we see that if $|f(z)| \leqq M$ in N, then

$$|P_n(z)| \leqq M \quad \text{for} \quad z \in N, \qquad n = 0, 1, 2, \cdots. \tag{1.7}$$

2. **Some uniqueness properties of analytic mappings.** H. Cartan [13] and C. Carathéodory [12] have proved two very elegant results on the uniqueness of analytic mappings. We shall give these two results here:

THEOREM 1 (CARTAN). *Let D be a domain containing the origin and let T be an analytic mapping of D into a bounded domain $D' \subset D$. If the linear part of T is the identity, that is, if T has the form*

$$T: \quad z_j' = f_j(z_1, \cdots, z_k) = z_j + (\textit{higher powers}), j = 1, \cdots, k, \tag{2.1}$$

then T is actually the identity

$$f_j(z) \equiv z_j, \qquad j = 1, \cdots, k. \tag{2.2}$$

PROOF. Expand the $f_j(z)$ in terms of homogeneous polynomials:

$$f_j(z) = z_j + P^j_{r_j}(z) + P^j_{r_j+1}(z) + \cdots \tag{2.3}$$

with r_j an integer not less than 2. In writing $f_j(z)$ in this form, we are assuming that the polynomials P^j_n of degree n for $1 < n < r_j$ all vanish identically. Now iterate the mapping. We find that T^2 has the expansion

$$T^2\colon\ z_j'' = z_j' + P^j_{r_j}(z') + \cdots = z_j + 2P^j_{r_j}(z) + \cdots, \tag{2.4}$$

and in general

$$T^m\colon\ z_j^{(m)} = z_j + mP^j_{r_j}(z) + \cdots. \tag{2.5}$$

Now T maps D into a bounded domain $D' \subset D$, hence the same is true of all the iterates $T^2, T^3, \cdots$. Thus the mapping functions

$$f_j^{(m)}(z) = z_j + mP^j_{r_j}(z) + \cdots \tag{2.6}$$

are uniformly bounded in D. By the result given at the end of §1, this implies that

$$\left| mP^j_{r_j}(z) \right| \leqq M, \qquad \text{for } j = 1, \cdots, k;\ m = 1, 2, \cdots,$$

for z in a sufficiently small neighborhood of the origin. This implies that $P^j_{r_j}(z) \equiv 0$ and hence (2.2) holds.

Carathéodory's uniqueness theorem follows almost immediately from Cartan's theorem.

THEOREM 2 (CARATHÉODORY). *There exists at most one* 1-1 *analytic mapping of a domain D onto a bounded domain D' which maps a point O of D onto a point O' of D' and whose first partial derivatives in O have prescribed values.*

PROOF. If S and T are two such mappings, then the mapping ST^{-1} is a mapping of D' onto itself and the hypotheses of Theorem 1 are satisfied. Thus

$$ST^{-1} = I \quad \text{or} \quad S = T.$$

An important corollary has been derived from Theorem 2. First we make a definition. A *circular* domain is a domain which is mapped onto itself by all the mappings

$$T(\theta)\colon\ z_j' = e^{i\theta} z_j, \qquad j = 1, \cdots, k;\ 0 \leqq \theta < 2\pi.$$

A circular domain is called *proper* if it contains the origin. Circular

domains were introduced by Behnke [1] and Carathéodory [11] and have been investigated by various writers. We shall return to them in §4. For the present we give only the following result.

COROLLARY. *Every automorphism of a proper bounded circular domain which leaves the origin fixed is a homogeneous linear transformation.*

PROOF. Let

$$A: \quad z_j' = \sum_{n=1}^{\infty} P_n^j(z_1, \cdots, z_k), \qquad j = 1, \cdots, k,$$

be any such automorphism. Then $T(\theta)A$ has the form

$$T(\theta)A: \quad z_j' = \sum_{n=1}^{\infty} e^{i\theta} P_n^j(z_1, \cdots, z_k),$$

and $AT(\theta)$ has the form

$$AT(\theta): \quad z_j' = \sum_{n=1}^{\infty} P_n^j(z_1 e^{i\theta}, \cdots, z_k e^{i\theta}) = \sum_{n=1}^{\infty} e^{in\theta} P_n^j(z).$$

Since D is circular and A is an automorphism, both transformations $T(\theta)A$ and $AT(\theta)$ are automorphisms of D, with the origin fixed. Also, both have the same linear parts. Hence, by Theorem 2, they are the same. This can happen, however, only if $P_n^j(z) \equiv 0$ for $n > 1$, that is only if A is linear.

The corollary has been proved by Behnke [2], Bergman [5], H. Cartan [13], and Welke [18]. Bergman's proof, which is by a different method, will be given in §5.

3. **Inner mappings and automorphisms with a fixed point.** Cartan's theorem tells us that if we have an inner analytic mapping T of a domain D containing the origin into a bounded domain D', and if the linear part of T is the identity, then T is actually the identity. Now suppose we have a general inner analytic mapping of a bounded domain D with the origin as a fixed point:

$$(3.1) \qquad T: \quad z_j' = a_{1j}z_1 + \cdots + a_{kj}z_k + \cdots, \qquad j = 1, \cdots, k.$$

Denoting by Δ_T the value of the functional determinant at the origin:

$$(3.2) \qquad \Delta_T = |a_{ij}|_{i,j=1 \cdots, k},$$

Cartan [14] and Carathéodory [12] have shown that

$$|\Delta_T| \leqq 1 \tag{3.3}$$

holds for every inner mapping, with the equality

$$|\Delta_T| = 1 \tag{3.4}$$

holding if and only if T is an automorphism. This result is very important in the theory of analytic mapping and will be useful to us in a later section.

The result has recently been generalized a little by Bochner and the speaker [10]. Denoting by $\lambda_1, \cdots, \lambda_k$ the characteristic roots of the matrix (a_{ij}), they have shown that

$$|\lambda_1| \leqq 1, \cdots, |\lambda_k| \leqq 1 \tag{3.5}$$

holds for every inner transformation (3.1) and

$$|\lambda_1| = \cdots = |\lambda_k| = 1 \tag{3.6}$$

holds if and only if T is an automorphism.

We shall not give the proofs of these results.

4. Groups of automorphisms of a bounded domain with a fixed point. Near the end of §2 we introduced the notion of a circular domain, a domain which with a point (z^0) contains also all the points

$$(z_1^0 e^{i\theta}, \cdots, z_k^0 e^{i\theta}), \qquad 0 \leqq \theta < 2\pi. \tag{4.1}$$

Cartan [13] has generalized this notion as follows: Let $(m_1, \cdots, m_k)$ be k integers (positive, negative, or zero). Then an $(m_1, \cdots, m_k)$-circular domain is one which, with a point (z^0), contains all the points

$$(z_1^0 e^{im_1\theta}, \cdots, z_k^0 e^{im_k\theta}), \qquad 0 \leqq \theta < 2\pi. \tag{4.2}$$

The domain is called *proper* if it contains the origin.

The domains of circular type play in several respects the role of the circle in one complex variable. It is, therefore, natural to question whether every bounded domain D can be mapped analytically onto some domain of circular type, or if not, which ones can. In the first place, an $(m_1, \cdots, m_k)$-circular domain admits an infinite number of automorphisms with the origin as a fixed point, namely those of (4.2). Therefore, if it is possible to map analytically a given domain D onto an $(m_1, \cdots, m_k)$-circular domain, the domain D must admit an infinite number of automorphisms leaving an interior point fixed. Cartan [13] (for $k=2$) has shown that this condition is also sufficient. Following the résumé given by Behnke and Thullen [4], we shall give a brief outline of Cartan's argument.

THEOREM 3 (CARTAN). *The group G of all automorphisms of a bounded domain D which leave the origin fixed ($O \in D$) is isomorphic with a compact group Γ of homogeneous linear transformations; the determinant of each transformation of Γ is of absolute value unity.*

By Carathéodory's theorem, Theorem 2, the group Γ consists merely of the linear parts of the members of G, and by (3.4) the determinant of each transformation of G is of absolute value unity.

An infinite compact group Γ of homogeneous linear transformations (in the space of two complex variables) always leaves invariant an Hermitian form

$$A w\bar{w} + Bz\bar{z} + Cw\bar{z} + \bar{C}\bar{w}z \tag{4.3}$$

(Weyl [19]). By a suitable linear transformation L this Hermitian form will be transformed into a normalized form

$$w\bar{w} + z\bar{z} \tag{4.4}$$

which is invariant under the corresponding group $\Gamma^* = L\Gamma L^{-1}$.

THEOREM 4 (Cartan). *Every infinite compact group of homogeneous linear transformations, which leave invariant the form* (4.4), *contains a subgroup of the form*

$$T_{m,p}(\theta):\quad w' = we^{im\theta},\qquad z' = ze^{ip\theta},\qquad 0 \leqq \theta < 2\pi. \tag{4.5}$$

Let G_0 be the subgroup of G which is isomorphic to the group $\{T_{m,p}(\theta)\}$. Denoting by $A(\theta)$ the isomorph in G of $T_{m,p}(\theta)$, we know that $A(\theta)$ must have the form

$$A(\theta):\quad \begin{aligned} w' &= f(w, z; \theta) = e^{im\theta}w + \cdots, \\ z' &= g(w, z; \theta) = e^{ip\theta}z + \cdots. \end{aligned} \tag{4.6}$$

Now define

$$\begin{aligned} F(w, z) &= \frac{1}{2\pi}\int_0^{2\pi} f(w, z; \theta)e^{-im\theta}d\theta, \\ G(w, z) &= \frac{1}{2\pi}\int_0^{2\pi} g(w, z; \theta)e^{-ip\theta}d\theta. \end{aligned} \tag{4.7}$$

Then Cartan showed that except possibly for certain exceptional cases the mapping

$$S:\quad w' = F(w, z),\qquad z' = G(w, z) \tag{4.8}$$

is a 1-1 analytic mapping of D onto a proper (m, p)-circular domain

with $SO=O$. (It is easily seen that the linear part of S is the identity.) In the exceptional cases, Cartan showed that S can be replaced by a mapping $\tilde{S}$ which has the requisite properties. Hence, we have:

THEOREM 5 (CARTAN). *If a bounded domain D admits an infinite group G of automorphisms with the origin as a fixed point $(O \in D)$, then D can be mapped in a* 1-1 *analytic manner on a bounded proper (m, p)-circular domain D' with the origin going into the origin.*

The structure of the group tells one what type of circular domain it will go into.

It is known that there exist bounded domains D which possess the following property: given any point P in D, the automorphisms of D which leave P fixed are finite in number [13]. Hence, *not every bounded domain can be mapped onto an (m, p)-circular domain.*

This concludes our résumé of Cartan's work. Certain aspects of it have been generalized. We shall indicate this generalization.

Let D be a bounded domain in E_{2k} containing the origin, and let G be a group of automorphisms of D with the origin as a fixed point. If $T(\alpha)$, $T(\beta)$, and so on, are elements of G, and if we write the expansion of $T(\alpha)$ in the form

$$T(\alpha): \quad z_i' = a_1^i(\alpha)z_1 + \cdots + a_k^i(\alpha)z_k + \sum_{n_1+\cdots+n_k \geqq 2} a_{n_1\cdots n_k}^i(\alpha) z_1^{n_1} \cdots z_k^{n_k}, \tag{4.9}$$

then Carathéodory's uniqueness theorem, Theorem 2, states that if

$$a_i^j(\alpha) = a_i^j(\beta), \qquad i, j = 1, \cdots, k, \tag{4.10}$$

then

$$a_{n_1\cdots n_k}^j(\alpha) = a_{n_1\cdots n_k}^j(\beta), \quad j = 1, \cdots, k; \; n_1 + \cdots + n_k \geqq 2. \tag{4.11}$$

Actually a little more than this has recently been proved [10], namely:

THEOREM 6 (BOCHNER AND MARTIN). *If $T(\alpha)$, $T(\beta)$, and so on, are elements of a group G of automorphisms of a bounded domain D with the origin as a fixed point, and if $(j, n_1, \cdots, n_k)$ is any fixed multi-index, then corresponding to any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\sum_{\rho,\sigma=1}^{k} \left| a_\sigma^\rho(\alpha) - a_\sigma^j(\beta) \right| \leqq \delta \tag{4.12}$$

implies

$$\left| a_{n_1\cdots n_k}^j(\alpha) - a_{n_1\cdots n_k}^j(\beta) \right| \leqq \epsilon. \tag{4.13}$$

(The a's are the coefficients in the expansion of the T's as in (4.9).)

The proof of this theorem uses strongly the uniqueness theorem of Carathéodory. We shall omit the proof. By use of Theorem 6, Bochner and the speaker [10] have obtained the following theorem which is related to Cartan's theorem, Theorem 5.

THEOREM 7. *If* $\{T(\alpha)\}$ *is a group G of automorphisms of a bounded domain D with the origin as a fixed point, then there exists a change of coordinates S in the neighborhood of the origin*

$$S: \quad z_i' = z_i + (\textit{higher powers}), \qquad j = 1, \cdots, k,$$

such that the given automorphisms are linear transformations in the new coordinates,

$$L(T(\alpha)) = ST(\alpha)S^{-1}.$$

We shall outline the proof of this theorem as given in [10].

By Theorem 2 each $T(\alpha)$ is uniquely determined by $L(T(\alpha))$ and hence by the k^2 complex numbers $a_\mu^j(\alpha)$, $\mu, j = 1, \cdots, k$. Thus, every element of G can be identified with a point in Euclidean space E of $2k^2$ dimensions, and G itself with a bounded point set in E. Also since

$$L(T(\alpha)\cdot T(\beta)) = L(T(\alpha))\cdot L(T(\beta)),$$

the group product $\beta\alpha$ of any two elements α, β of G is given by

$$a_m^j(\beta\alpha) = \sum_{\mu=1}^{k} a_\mu^j(\beta)a_m^\mu(\alpha), \qquad j, m = 1, \cdots, k.$$

Thus in the ordinary Euclidean topology of E, the group product $\beta\alpha$ is a continuous function of β and α.

Now, since G is bounded, there exists an additive measure μ on G with the following properties among others:

(i) the measure of the total set G is 1,

(ii) the measure is group invariant, and

(iii) every function $f(\alpha)$ on G which is uniformly continuous on G (in the topology of E) is integrable.

Now Theorem 6 states that each coefficient of $T(\alpha)$ is uniformly continuous on G. Hence, by property (iii) each coefficient of $T(\alpha)$ is integrable relative to our measure. It follows that the transformation $L(T(\alpha^{-1}))\cdot T(\alpha)$ is measurable and it can be shown that the desired change of coordinates S is given by

$$S = \int_G L(T(\alpha^{-1})\cdot T(\alpha)d\mu(\alpha).$$

5. Bergman's representative domains. In connection with the second problem of the theory of analytic mapping mentioned in the introduction, Bergman [5] has introduced the notion of a *representative domain* of a class of *equivalent domains*, and in a series of papers he has developed several interesting aspects of the theory. A portion of his work has been simplified by Welke [18] and parts of it have been summarized by Behnke and Thullen [4] and by Bochner and the speaker [10]. In this section we shall follow the latter summary for the most part.

First we give two definitions due to Bergman.

Normalized mapping—An analytic mapping of a domain D containing the origin is called normalized (with respect to the origin) if its linear part is the identity.

Equivalent domains—Two domains D and D' are called equivalent if they can be mapped onto each other by a 1-1 analytic normalized mapping.

We shall use these two notions later in this section.

Let D be a domain in E_{2k} and denote by $\mathfrak{L}_2$ the set of all functions $f(z_1, \cdots, z_k)$ analytic in D and such that the norm

$$\|f\| = \left[\int_D |f(z)|^2 d\omega_z\right]^{1/2} \tag{5.1}$$

is finite ($d\omega_z = dx_1 dy_1 \cdots dx_k dy_k$).

If D has finite volume then every bounded analytic function in D belongs to $\mathfrak{L}_2$. If D is also bounded this includes all polynomials. For the sake of simplicity, we restrict ourselves to bounded domains D. The space $\mathfrak{L}_2$ is then a Hilbert space.

Now take a point in D, call it the origin and expand all functions $f(z)$ of $\mathfrak{L}_2$ in power series about the origin. It can be shown that there exists a unique minimal function

$$f_0(z) = 1 + \text{(higher powers)} \tag{5.2}$$

such that

$$\|f_0\| < \|f\| \tag{5.3}$$

for every f of $\mathfrak{L}_2$ with the development

$$f(z) = 1 + \text{(higher powers)}. \tag{5.4}$$

The same is true for functions of the form

$$f_j(z) = z_j + \text{(higher powers)}, \tag{5.5}$$

that is for each $j=1, \cdots, k$ there exists a unique function $f_j(z)$ of $\mathcal{L}_2$ for which

$$\|f_j\| < \|f\| \tag{5.6}$$

holds for every $f \in \mathcal{L}_2$ in D with the development

$$f(z) = z_j + \text{(higher powers)}. \tag{5.7}$$

Bergman has shown that these $(k+1)$-minimal functions possess some beautiful forms of invariance. Consider a 1-1 normalized analytic transformation

$$z_j = z_j(z_1', \cdots, z_k') = z_j' + \text{(higher powers)} \tag{5.8}$$

from D onto a domain D' with nonvanishing Jacobian

$$J(z') = \partial(z_1, \cdots, z_k)/\partial(z_1', \cdots, z_k'). \tag{5.9}$$

Now, by the ordinary transformation of variables

$$\int_D |f(z)|^2 d\omega_z = \int_{D'} |f(z(z'))|^2 |J(z')|^2 d\omega_{z'}, \tag{5.10}$$

and it is not hard to see that

$$f(z) \leftrightarrow f(z(z'))J(z') \tag{5.11}$$

is a 1-1 transformation of $\mathcal{L}_2(z)$ onto $\mathcal{L}_2(z')$. Also since

$$J(z') = 1 + \text{(higher powers)} \tag{5.12}$$

we have

$$f_j^D(z(z')) \equiv f_j^{D'}(z')J(z'), \qquad j = 0, 1, \cdots, k, \tag{5.13}$$

where $f_j^D(z)$ and $f_j^{D'}(z')$ are the minimal functions for D and D' respectively. Hence at least in a neighborhood of the origin

$$f_j^D(z(z'))/f_0^D(z(z')) \equiv f_j^{D'}(z')/f_0^{D'}(z'), \qquad j = 1, \cdots, k, \tag{5.14}$$

and thus these quotients are absolute invariants. Omitting the superscript D, we denote these functions by $w_1(z), \cdots, w_k(z)$ and we observe that they have the expansion

$$w_j(z) = z_j + \text{(higher powers)}, \qquad j = 1, \cdots, k. \tag{5.15}$$

Now assume that our domain D is such that for D these k functions have the special form

$$w_j(z) \equiv z_j, \qquad j = 1, \cdots, k. \tag{5.16}$$

Transcribe them by (5.8) onto D'. Being invariants they go over into

$$w_j^*(z') = z_j(z_1', \cdots, z_k') = z_j' + \text{(higher powers)}. \tag{5.17}$$

Hence they cannot have the specialized form (5.16) in the domain D' unless the transformation (5.8) is the identity $z_j \equiv z_j'$. In other words the specialized form (5.16) cannot occur for more than one domain D in a class of equivalent domains and thus the specialized form (5.16) singles out the domain D from all other domains D' which are equivalent to D under a 1-1 analytic mapping which preserves the origin. This specialized domain is called the *representative* of its class by Bergman.

Now let K be a class of equivalent domains and assume that the class has a representative domain R. Then each domain D of K can be mapped onto R in a 1-1 manner by a normalized mapping. Denote this mapping by

$$T: \quad z_j' = g_j(z), \qquad z \in D, \tag{5.18}$$

and consider the mapping

$$S: \quad z_j' = w_j^D(z) \tag{5.19}$$

where the $w_j^D(z)$ are the invariant quotients for D. We do not assume that S is 1-1. But by the invariance we know that

$$w_j^D[g_1^{-1}(z'), \cdots, g_k^{-1}(z')] \equiv w_j^R(z'), \tag{5.20}$$

and since R is representative

$$w_j^R(z') \equiv z_j', \qquad j = 1, \cdots, k.$$

Hence

$$w_j^D[g_1^{-1}(z'), \cdots, g_k^{-1}(z')] \equiv z_j', \tag{5.21}$$

that is

$$w_j^D(z) \equiv g_j(z). \tag{5.22}$$

Thus S gives a 1-1 normalized mapping of D onto R.

This yields a very beautiful result due to Bergman.

THEOREM 8 (BERGMAN). *Let K be a class of equivalent domains, equivalent with respect to* 1-1 *normalized analytic mappings. For $D \in K$ let*

$$f_0^D(z) = 1 + (\textit{higher powers}), \tag{5.23}$$

(5.24) $$f_j^D(z) = z_j + (\textit{higher powers}), \qquad j = 1, \cdots, k,$$

be the $k+1$ *(unique) minimal functions for* D *and define*

(5.25) $$w_j^D(z) = f_j^D(z)/f_0^D(z), \qquad j = 1, \cdots, k.$$

There exists at most one domain R *of* K *for which the functions* $w_j^R(z)$ *have the specialized form*

$$w_j^R(z) \equiv z_j, \qquad j = 1, \cdots, k.$$

If there is actually such a domain R *then every domain* D *of* K *can be mapped onto* R *by a* 1-1 *normalized analytic mapping, and the mapping which does this is given by*

(5.26) $$z_j' = w_j^D(z), \qquad j = 1, \cdots, k; z \in D.$$

It can easily be seen that there is a representative domain if for at least one domain D of K the mapping

$$z_j' = w_j^D(z)$$

is 1-1.

Bergman has observed that every bounded proper circular domain Δ is a representative domain. To see this write the minimal function $f_0(z)$ for a bounded proper circular domain Δ in the form

(5.27) $$f_0(z) = 1 + P_1(z) + P_2(z) + \cdots,$$

where $P_n(z)$ is a homogeneous polynominal of degree n. Then

(5.28) $$f_0(z_1e^{i\theta}, \cdots, z_ke^{i\theta}) = 1 + e^{i\theta}P_1(z) + e^{2i\theta}P_2(z) + \cdots.$$

Now the volume element $d\omega_z$ is invariant under the circular transformation

$$z_j \leftrightarrow z_je^{i\theta}.$$

Hence the function (5.28) has the same norm as (5.27). Hence both are minimal. But since the development of each starts with 1 and since the minimal functions are unique, (5.27) and (5.28) must be identical. Thus

$$P_n(z) \equiv 0, \qquad n = 1, 2, 3, \cdots,$$

and

$$f_0(z) \equiv 1.$$

In a similar manner we see that

$$f_j(z) \equiv z_j, \qquad j = 1, \cdots, k,$$

and thus

$$w_j(z) \equiv z_j.$$

This yields the desired result.

Now suppose we have a bounded domain D which is equivalent to a bounded proper circular domain Δ. On applying Bergman's theorem, Theorem 8, we have a procedure for mapping D onto Δ; we merely calculate the $k+1$ minimal functions

$$f_\mu(z), \qquad \mu = 0, 1, \cdots, k,$$

for D and map by

$$z_j' = f_j(z)/f_0(z), \qquad j = 1, \cdots, k.$$

By use of complex orthonormal functions Bergman [6] has simplified the calculation of the minimal functions.

A second important consequence of this result has been derived by Bergman and Welke, namely the corollary of §2.

Every origin preserving automorphism of a bounded circular domain Γ containing the origin is a homogeneous linear transformation.

PROOF. Let

$$T: \quad z_j = a_{1j}z_1' + \cdots + a_{kj}z_k' + \cdots$$

be an automorphism of Δ. It can be normalized by a homogeneous linear transformation L, and this homogeneous linear transformation carries Δ into a circular domain $\Delta' = L\Delta$. Now

$$TL: \quad z_j = z_j' + (\text{higher powers}), \qquad j = 1, \cdots, k,$$

is an automorphism of Δ'. Since Δ' is circular it is representative and hence the only 1-1 normalized automorphism of Δ' is the identity, that is $TL = I$ or $T = L^{-1}$. Thus T is linear.

6. **Mappings of unbounded domains.** In this section I want to give briefly and without proof a few results on mappings of unbounded domains. First we shall recall Cartan's theorem, Theorem 1, which states that if we have an inner (analytic) mapping T of a domain D containing the origin onto a bounded domain D', and if the linear part of T is the identity, then T is actually the identity.

This theorem does not hold if D' is unbounded. In fact, Bieberbach [9], using a result of Fatou [15, 16], has shown that there exist two entire functions $z' = z + (\text{higher powers})$, $w' = w + (\text{higher powers})$,

which map the entire space E_4 onto a proper subset of E_4 in a topological manner. Thus they define an inner mapping whose linear part is the identity and yet the mapping is not the identity.

In spite of this Behnke and Peschl [3] have been able to prove a very beautiful generalization of Cartan's theorem for certain unbounded domains.

THEOREM 9 (BEHNKE AND PESCHL). *Let D be a domain containing the origin and let D have the following property: There exist k bounded analytic functions*

$$g_j(z) = \sum_{n=n_j}^{\infty} P_n^j(z), \qquad j = 1, \cdots, k, \tag{6.1}$$

in D such that the determinant

$$J(z) = \partial(P_{n_1}^1, \cdots, P_{n_k}^k)/\partial(z_1, \cdots, z_k) \tag{6.2}$$

is not identically zero. Then every inner analytic mapping of D whose linear part is the identity is actually the identity.

The development (6.1) is the development of $g_j(z)$ in a series of homogeneous polynomials; there is no restriction on the value of the lowest degree n_j.

Using the result of Behnke and Peschl, Bochner and the speaker [10] have carried over part of the results of §3 to a class of unbounded domains. Their result is: *If D is not bounded but possesses the alternative property indicated in Theorem 9, if T is an inner mapping of D with the origin as a fixed point, if (3.4) holds, and if the linear parts $\{L_T, L_{T^2}, L_{T^3}, \cdots\}$ are uniformly bounded in D, then T is an automorphism.*

7. **Concluding remarks.** The field of analytic mappings is wide and contains numerous results. It has been impossible to give or even to summarize all of these results; as a consequence many interesting and important results have been omitted. Many of these will be found in the résumé of mapping contained in chapter 7 of the tract by Behnke and Thullen [4]. Several are given in the book by Bergman [6]. I would like to refer the listener to these two books for excellent treatments of various aspects of the theory.

Among the more recent papers on mapping I would like to mention two by Bergman and Spencer [7, 8] which have appeared since the publication of the books just mentioned. These papers point in a somewhat different direction from those considered in this talk, and they contain references to other recent treatments.

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SYRACUSE UNIVERSITY

THE OCTOBER MEETING IN NEW YORK

The three hundred ninety-ninth meeting of the American Mathematical Society was held at Columbia University on Saturday, October 30, 1943. The attendance included the following one hundred thirty-one members of the Society:

C. F. Adler, A. A. Albert, R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, E. G. Begle, Stefan Bergman, Felix Bernstein, Lipman Bers, J. H. Bigelow, Gertrude Blanch, A. H. Bowker, C. B. Boyer, A. B. Brown, R. C. Buck, Hobart Bushey, J. H. Bushey, S. S. Cairns, I. S. Carroll, L. M. Comer, T. F. Cope, Richard Courant, A. P. Cowgill, A. T. Craig, M. D. Darkow, R. L. Dietzold, Jesse Douglas, Jacques Dutka, Samuel Eilenberg, E. E. Fedder, J. M. Feld, A. D. Fialkow, Tomlinson Fort, R. M. Foster, W. C. G. Fraser, K. O. Friedrichs, R. E. Fullerton, Godofredo García, H. P. Geiringer, Abe Gelbart, David Gilbarg, B. P. Gill, Lewis Greenwald, Laura Guggenbühl, E. J. Gumbel, Jacques Hadamard, P. R. Halmos, Olaf Helmer, M. R. Hestenes, L. S. Hill, Einar Hille, Abraham Hillman, T. R. Hollcroft, Harold Hotelling, J. L. Howell, E. M. Hull, L. C. Hutchinson, W. H. Ingram, S. A. Joffe, S. A. Kiss, H. S. Kieval, J. R. Kline, B. O. Koopman, M. E. Ladue, Cornelius Lanczos, Solomon Lefschetz, B. A. Lengyel, Norman Levinson, D. C. Lewis, E. R. Lorch, A. N. Lowan, E. J. McShane, L. A. MacColl, H. F. MacNeish, A. J. Maria, D. H. Maria, W. T. Martin, L. W. Miller, E. C. Molina, Deane Montgomery, E. J. Moulton, G. W. Mullins, F. J. Murray, C. A. Nelson, G. E. Noether, P. B. Norman, C. O. Oakley, F. W. Perkins, George Piranian, E. L. Post, G. B. Price, L. L. Rauch, H. W. Reddick, Eric Reissner, Moses Richardson, J. F. Ritt, E. K. Ritter, S. L. Robinson, P. C. Rosenbloom, J. E. Rosenthal, M. G. Scherberg, O. H. Schmidt, Seymour Sherman, Max Shiffman, J. A. Shohat, James Singer, D. M. Smiley, M. F. Smiley, P. A. Smith, Wolfgang Sternberg, J. J. Stoker, R. R. Stoll, E. G. Straus, W. C. Strodt, Fred Supnick, Feodor Theilheimer, W. J. Thron, Peter Treuenfels, H. F. Tuan, J. W. Tukey, D. F. Votaw, W. R. Wasow, Louis Weisner, A. P. Wheeler, W. F. Whitmore, S. S. Wilks, John Williamson, Jacob Wolfowitz, Alexander Wundheiler, Oscar Zariski.

There was a session for contributed papers in the morning at which Professor Deane Montgomery presided. Professor J. F. Ritt presided at the afternoon session.

A business meeting was held at the opening of the afternoon session at which the Society voted to amend Article III, Section 1, and Article IX, Sections 1 and 2 of the By-Laws to read:

Article III. Editorial Committees

Section 1. There shall be six Editorial Committees—one of four members for the Bulletin, one of three members for the Transactions, one of three members for the Mathematical Reviews, one of three members for the Colloquium Publications, one of three members for the Mathematical Surveys, and one consisting of three representatives of the Society on the Board of Editors of the American Journal of Mathematics.

ARTICLE IX. PUBLICATIONS

Section 1. The Society shall publish an official organ called the Bulletin of the American Mathematical Society. It shall publish a journal called the Transactions of the American Mathematical Society, the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called Mathematical Reviews containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called Colloquium Publications which shall embody in book form new mathematical developments. It shall publish a series of monographs called Mathematical Surveys which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also cooperate in the conduct of the American Journal of Mathematics.

Section 2. The editorial management of the Bulletin, Transactions, Mathematical Reviews, Colloquium Publications, and Mathematical Surveys, and the participation of the Society in the management of the American Journal of Mathematics shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

Following the business meeting Professor W. T. Martin of Syracuse University gave an address entitled *Mappings by means of systems of analytic functions of several complex variables.*

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–8 were read in the morning session and papers 9–22 were presented by title. Paper 2 was read by Mr. Buck, paper 3 by Dr. Bers; paper 4 was read for the authors by Professor Jesse Douglas. Mr. Haimo was introduced by Professor F. L. Griffin, and Mr. Salzer by Professor A. N. Lowan.

1. Max Shiffman: *Isoperimetric inequalities and continuity of area for classes of surfaces.* (Abstract 49-11-272.)

2. R. C. Buck and Harry Pollard: *Convergence and summability properties of subsequences.* (Abstract 49-11-256.)

3. Lipman Bers and Abe Gelbart: *On functions satisfying certain systems of partial differential equations.* (Abstract 50-1-20.)

4. Edward Kasner and John DeCicco: *A generalized theory of contact transformations.* (Abstract 49-9-235.)

5. J. A. Shohat: *Parseval formula in its application to Van der Pol's and generalized equations.* (Abstract 49-11-293.)

6. Norman Levinson: *On a nonlinear differential equation of the second order.* (Abstract 49-11-264.)

7. R. R. Stoll: *Representations of completely simple semigroups.* (Abstract 49-11-248.)

8. L. A. MacColl: *Geometrical characterizations of some families of dynamical trajectories.* (Abstract 49-11-299.)

9. Stefan Bergman: *The determination of singularities of functions*

satisfying a partial differential equation from the coefficients of their series development. (Abstract 50-1-18-*t*.)

10. Lipman Bers and Abe Gelbart: *On a class of functions defined by certain partial differential equations.* (Abstract 50-1-19-*t*.)
11. R. C. Buck: *A note on subsequences.* (Abstract 49-11-254-*t*.)
12. R. C. Buck: *Multiple sequences.* (Abstract 49-11-255-*t*.)
13. R. C. Buck: *Partition of space.* (Abstract 49-11-296-*t*.)
14. Herbert Federer: *Surface area.* I. (Abstract 49-11-259-*t*.)
15. Herbert Federer: *Surface area.* II. (Abstract 49-11-260-*t*.)
16. Franklin Haimo: *Periodic functions on algebraic systems.* (Abstract 49-1-2-*t*.)
17. A. N. Lowan and H. E. Salzer: *Table of coefficients for inverse interpolation.* (Abstract 49-9-224-*t*.)
18. A. P. Morse: *A theory of covering and differentiation.* (Abstract 49-11-266-*t*.)
19. A. P. Morse and J. F. Randolph: *The ϕ rectifiable subsets of the plane.* (Abstract 49-11-267-*t*.)
20. F. J. Murray: *On the existence of quasi-complements in Banach spaces.* (Abstract 49-11-268-*t*.)
21. H. E. Salzer: *Coefficients for numerical integration with central differences.* (Abstract 49-9-237-*t*.)
22. Antoni Zygmund: *On certain integrals.* (Abstract 49-11-276-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE ANNUAL MEETING OF THE SOCIETY

The fiftieth Annual Meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, Friday and Saturday, November 26 and 27, 1943, in conjunction with meetings of the Mathematical Association of America. Over two hundred persons registered including the following one hundred sixty-five members of the Society:

R. P. Agnew, A. A. Albert, C. B. Allendoerfer, B. H. Arnold, Max Astrachan, W. L. Ayres, R. W. Babcock, Reinhold Baer, R. M. Ballard, Harry Bateman, J. C. Bell, S. F. Bibb, B. H. Bissinger, L. M. Blumenthal, J. L. Brenner, R. W. Brink, J. O. Brown, R. H. Bruck, E. L. Buell, H. E. Burns, Herbert Busemann, Albert Cahn, W. D. Cairns, W. B. Carver, W. B. Caton, W. F. Cheney, E. W. Chittenden, R. V. Churchill, A. B. Coble, L. W. Cohen, N. B. Conkwright, A. H. Copeland, J. J. Corliss, D. R. Curtiss, M. M. Day, John DeCicco, J. J. Dennis, Jesse Douglas, Arnold Dresden, D. M. Dribin, Samuel Eilenberg, Wade Ellis, Paul Erdös, H. J. Ettlinger, H. P. Evans, H. S. Everett, G. M. Ewing, J. V. Finch, N. J. Fine, Edward Fleisher, L. R. Ford, R. H. Fox, J. S. Frame, Morris Friedman, T. C. Fry, J. H. Giese, J. W. Givens, Michael Golomb, Cornelius Gouwens, L. M. Graves, V. G. Grove, D. W. Hall, P. R. Halmos, R. W. Hamming, W. L. Hart, C. T. Hazard, M. H. Heins, E. D. Hellinger, Edward Helly, Fritz Herzog, E. H. C. Hildebrandt, T. H. Hildebrandt, D. L. Holl, A. S. Householder, H. K. Hughes, Ralph Hull, M. G. Humphreys, Mildred Hunt, H. D. Huskey, M. H. Ingraham, B. W. Jones, William Karush, Robert Kates, D. E. Kearney, I. F. Keeler, Fred Kiokemeister, J. R. Kline, L. A. Knowler, W. C. Krathwohl, E. P. Lane, R. E. Langer, E. H. Larguier, C. G. Latimer, D. H. Leavens, R. H. Lence, A. L. Lewis, M. I. Logsdon, A. T. Lonseth, Eugene Lukacs, Janet MacDonald, C. C. MacDuffee, Morris Marden, A. V. Martin, W. T. Martin, Karl Menger, E. J. Mickle, H. J. Miser, W. L. Mitchell, C. W. Moran, Marston Morse, E. J. Moulton, S. B. Myers, A. L. Nelson, C. V. Newsom, K. L. Nielsen, Ivan Niven, E. A. Nordhaus, E. P. Northrop, C. D. Olds, Isaac Opatowski, W. V. Parker, B. C. Patterson, Tibor Radó, G. Y. Rainich, R. B. Rasmusen, W. T. Reid, Haim Reingold, W. H. Roever, R. G. Sanger, A. T. Schafer, R. D. Schafer, O. F. G. Schilling, G. E. Schweigert, M. E. Shanks, H. A. Simmons, G. W. Smith, I. S. Sokolnikoff, C. F. Stephens, N. E. Steenrod, A. H. Stone, D. M. Stone, M. H. Stone, E. B. Stouffer, R. L. Swain, J. L. Synge, R. M. Thrall, W. J. Trjitzinsky, S. M. Ulam, H. S. Wall, S. E. Warschawski, Warren Weaver, K. W. Wegner, J. V. Wehausen, C. P. Wells, E. T. Welmers, M. E. Wescott, M. D. Wetzel, G. W. Whitehead, G. T. Whyburn, L. R. Wilcox, R. L. Wilder, K. P. Williams, R. S. Wolfe, J. W. T. Youngs, Daniel Zelinsky.

Sessions for the reading of contributed papers were held Friday at 9:30 A.M. and 2:00 P.M. and Saturday at 10:45 A.M. and 3:15 P.M. The presiding officers for these sessions were Professors L. R. Ford, R. E. Langer, R. L. Wilder, and Dean M. H. Ingraham.

Friday morning following the session for contributed papers, Professor Reinhold Baer of the University of Illinois delivered an address

entitled *The higher commutator subgroups*. Professor A. A. Albert presided for this address.

Friday evening Professor Harry Bateman delivered the seventeenth Josiah Willard Gibbs Lecture on *The control of elastic fluids*. President M. H. Stone presided.

The annual business meeting and election of officers was held on Saturday morning, President Stone presiding. At the close of the business session President Stone announced the sixth award of the Bôcher Memorial Prize to Professor Jesse Douglas of Brooklyn College for his memoirs:

(a) *Green's function and the problem of Plateau*, American Journal of Mathematics, volume 61 (1939) pp. 545–589.
The most general form of the problem of Plateau, American Journal of Mathematics, volume 61 (1939) pp. 590–609.

(b) *Solution of the inverse problem of the calculus of variations*, Transactions of the American Mathematical Society, volume 50 (1941) pp. 71–128.

Professor Douglas then gave a résumé of his prize-winning papers.

Saturday afternoon with President Stone presiding, Professor Marston Morse of the Institute for Advanced Study delivered the Retiring Presidential Address entitled *New settings for topology in analysis*.

Sessions of the Mathematical Association of America were held on Saturday evening and Sunday morning. At the business meeting of the Association on Sunday morning Professors C. V. Newsom and G. M. Ewing presented a resolution of appreciation to the officials of the Museum of Science and Industry and the local committee on arrangements on behalf of the two mathematical organizations.

At the meeting of the Board of Trustees at 6:00 P.M., November 26, 1943, in the Hotel Windermere East, there was no quorum present and the Board adjourned to January 8, 1944. The Council met at 9:30 P.M. on November 26, 1943, in the Hotel Windermere East.

The Secretary announced the election of the following twelve persons to membership in the Society:

Mr. R. H. Bing, University of Texas;
Mr. Patrik Butler, University of Texas;
Dr. Shiing-shen Chern, Institute for Advanced Study;
Professor Russell Dunholter, College of Engineering and Commerce, University of Cincinnati;
Professor William Henry Fagerstrom, College of the City of New York;
Professor Alois Joseph Lorenz, St. Michael's College, Winooski Park, Vt.;
Dr. Earl John Mickle, Ohio State University;

Mr. Calvin Northrup Mooers, Naval Ordinance Laboratory, Washington, D. C.;
Dr. Chaim Leib Pekeris, Massachusetts Institute of Technology;
Professor Louis Wright Roberts, Howard University;
Dr. John Daniel Rommel, Jr., U.S.N.R., San Diego, Calif.;
Dr. Gail Sellers Young, Jr., Purdue University.

It was reported that the following had been elected as nominees on the institutional memberships of the institutions indicated:

Columbia University: Messrs. Robert Nathan Ascher, Edmund Churchill, and Huseyin Demir, Miss Aida Kalish, Messrs. Howard Levene, Robert King Ritt, and Cengiz Uluçay.
Michigan State College: Mr. Lawrence Duncan Childs.
University of Minnesota: Mr. Henry Damon Colson.
Ohio State University: Dr. Leslie Haynes Miller, Mr. Herbert Charles Parrish.
University of Pennsylvania: Mr. Richard Crowley Campbell.

The following appointments by Acting President C. C. MacDuffee were reported: as an auditor of the Society's accounts for 1943, Professor T. F. Cope; as tellers for the election at the 1943 Annual Meeting, Professors W. C. Krathwohl and E. R. Lorch, Dr. W. C. Strodt; as an editorial committee for Mathematical Surveys, to serve until December 31, 1944, Professors F. D. Murnaghan (Chairman), A. A. Albert, and T. H. Hildebrandt; as a Committee on the Award of the Cole Prize in Algebra, to be awarded at the 1944 Annual Meeting for papers published during the period 1939–1943, Professors E. T. Bell (Chairman), A. A. Albert, and N. H. McCoy; as a member of the Committee on Places of Meetings, for a period of three years beginning 1944, Professor C. R. Adams (committee now consists of Professors W. B. Carver, Chairman, C. R. Adams, and E. W. Chittenden); as new members of the Committees to Select Hour Speakers: for Annual and Summer Meetings, Professor J. D. Tamarkin (committee now consists of Professors J. R. Kline, Chairman, J. D. Tamarkin, and R. L. Wilder); for Eastern Sectional Meetings, Professor E. J. McShane (committee now consists of Professors T. R. Hollcroft, Chairman, E. J. McShane, and Saunders MacLane); for Western Sectional Meetings, Professor A. A. Albert (committee now consists of Professors W. L. Ayres, Chairman, A. A. Albert, and R. E. Langer); for Far Western Sectional Meetings, Professor Harry Bateman (committee now consists of Professors A. D. Michal, Chairman, Harry Bateman, and W. M. Whyburn).

Times and places of meetings during 1944 were set as follows: April 29 in Berkeley, California; April 28–29 in Chicago, Illinois; April 28–29 in New York City.

Certain invitations to give hour addresses were announced: Pro-

fessor E. L. Post for the February, 1944, meeting in New York City; Professors André Weil and E. R. Lorch for the April, 1944, meeting in New York City; Professor C. C. MacDuffee for the 1944 Summer Meeting.

The Secretary reported the resignations of Professors Saunders MacLane and G. B. Price from the Committee on Publicity (committee now consists of Professors R. M. Foster, Chairman, J. B. Rosser, and Dr. J. M. Thompson).

The Secretary reported that Professor Tibor Radó had accepted the invitation of the Council to deliver a set of Colloquium Lectures in 1945 and that the title of the Lectures will be *Length and area*.

The Council voted to invite Professor John von Neumann to deliver the eighteenth Josiah Willard Gibbs Lecture at the 1944 Annual Meeting.

The Secretary reported that the ordinary membership in the Society is now 2594, including 208 nominees of institutional members and 67 life members. There are also 87 institutional members. The total attendance at all meetings in 1943 was 1,028; the number of papers read was 278; there were 12 hour addresses, 6 symposium addresses, 3 Colloquium Lectures, 1 Gibbs Lecture, and a Retiring Presidential Address; the number of members attending at least one meeting was 619. The Secretary also reported the completion of the biennial membership campaign held during November when approximately 3,600 college teachers of mathematics, including 560 members of the Mathematical Association of America, were invited to membership in the Society.

At the annual election which closed on November 27, and at which 486 votes were cast, the following officers were elected:

Vice Presidents, Professors Einar Hille and G. T. Whyburn.

Associate Secretaries, Professors W. L. Ayres, A. D. Michal, and G. B. Price.

Member of the Editorial Committee of the Bulletin, Professor R. E. Langer.

Member of the Editorial Committee of the Transactions, Professor E. J. McShane.

Member of the Editorial Committee of the Colloquium Publications, Professor G. T. Whyburn.

Member of the Editorial Committee of Mathematical Reviews, Professor J. D. Tamarkin.

Member of the Editorial Committee of the American Journal of Mathematics, Professor Hassler Whitney.

Members-at-large of the Council, Professors Harry Bateman, Garrett

Birkhoff, M. R. Hestenes, Harold Hotelling, Nathan Jacobson, R. D. James.

Professor J. W. T. Youngs of Purdue University was appointed Acting Associate Secretary to serve during the time that Associate Secretary G. B. Price is engaged in work for the government.

The Council adopted the following resolution concerning the work of Associate Secretary M. H. Ingraham in connection with the financial affairs of the Society:

With the retirement of Dean M. H. Ingraham from an Associate Secretaryship in the American Mathematical Society, the Council wishes to place on record an expression of its profound appreciation of his services rendered to the Society and to the cause of mathematics.

In the early nineteen thirties with at best a static income and with a rapidly expanding program of research publications, the responsible officers of the Society faced a problem of unusual difficulty. Various temporary expedients had been tried but the fact remained that either new funds of considerable magnitude must be tapped or the size of the journals drastically cut. Secretary Ingraham was at that time in large part responsible for putting into successful practice the scheme of individual contributing memberships by which members made voluntary increases in dues. But it soon became apparent that these extra funds were not sufficient to meet the situation. With the argument that the mathematicians had themselves made a real contribution to the balancing of the budget, it seemed possible to seek the financial backing of the universities. Professor Ingraham, one of the prime movers in this fruitful idea, was selected to interview university authorities. The Rockefeller Foundation, which over a period of years had been generous to the Society, was willing to pay the expenses of an extended trip among the mathematical centers.

Secretary Ingraham was remarkably successful not only in enlisting the support of the universities but also in leaving behind a friendlier and more understanding attitude toward research in mathematics. In each of the intervening years he kept in touch with the institutions which subscribed and, as a result, a sum of seventy-five hundred dollars has flowed annually into the treasury, thus enabling the publication program to be kept at a high level. And in a variety of other ways the Society has benefited from his leadership.

Perhaps no member of the Society is more widely known among his colleagues or has a larger circle of friends. The Council desires to extend hearty thanks on behalf of the Society and to wish him every success in his new duties at the University of Wisconsin.

Since the 1943 Annual Meeting was held before the close of the Society's fiscal year, it was impossible to present a full report of the Treasurer. (Excerpts from the final report of the Treasurer however appear on pages 29–32 of this Bulletin.) For the information of the Council a general statement on the current finances of the Society was presented. It was reported: (1) that total receipts for the fiscal year would exceed estimates and that the net excess of receipts over disbursements would probably be about $5,000; (2) that income from individual and from institutional memberships would be at least as

great as in 1942; (3) that the income from the sale of the Transactions would exceed estimates by nearly $1,000, principally because of an unusual demand for back volumes; (4) that during 1943 the Society had received a full year's income, amounting to almost $4,000, from the estate of Robert Henderson, a grant of $2,500 from the Rockefeller Foundation for the use of the War Policy Committee, and the final payment on the principal of the Reilly Estate (the total receipts from the Reilly Estate amount to $23,651.47 and have been added to the pooled investments of the Society).

The Librarian reported the following additions to the library: 41 books, 81 bound volumes of periodicals, 7 pamphlets, 306 dissertations in printed form, 149 dissertations on microfilm and 1 volume on microfilm. He reported that during the year considerable progress had been made on the project of securing copies of all American doctoral dissertations for the library. From the gift to the library by Professor R. C. Archibald, former Librarian of the Society, a microfilm reader has been purchased which is installed in the reading room and is available for use by members of the Society.

The Bulletin Editorial Committee reported that it would publish approximately 1,000 pages in the 1943 volume and that, because the present supply of unpublished manuscripts is very small, prompt publication is now possible. The editors also pointed out that, while shorter papers are desired, there is no longer an adherence to any strict rule of length.

The Transactions Editorial Committee reported that it expected to publish about 1,065 pages in 1943 and that, in spite of the fact that there is no present backlog of papers, at least Volume 55 would be of normal size. It was reported that Professor A. A. Albert would be Managing Editor during 1944–1945 and that Professors G. A. Hedlund, Nathan Jacobson, W. T. Reid, N. E. Steenrod, and Antoni Zygmund had been appointed Associate Editors, to replace Professors T. H. Hildebrandt, R. E. Langer, H. P. Robertson, and Gabor Szegö. The retirement of Professor Einar Hille from the Editorial Committee of the Transactions marks the end of an unusual record of twenty-one years of continuous connection with the editorial work of this journal. The Society has been fortunate to have had his wise counsel and devoted service for this long period.

The Mathematical Reviews Editorial Committee reported that there had been a marked decrease in the amount of mathematical material published throughout the world, that the editors had been able to obtain material from Russia regularly, and that an almost

complete supply of material published in Germany up to the beginning of 1943 had been secured.

The Council voted to place Mathematical Surveys on the same basis as the Colloquium Series in the regulations concerning special privileges for enlisted men. The regulations as amended at the Summer and Annual Meetings were distributed to all members of the Society on January 1.

President Stone reported to the Council that the War Policy Committee is actively studying problems concerning the proper use of mathematicians in the war. *Department chairman are urgently requested to inform the Secretary concerning cases in which the new deferment procedures lead to the improper classification of a mathematician.* (The new procedures were described in a memorandum published in the November issue of this Bulletin.) It was further reported that an active subcommittee of the War Policy Committee (Professor W. L. Hart, Chairman) is studying the mathematical aspects of the various war training programs. A report was presented by the Committee on Available Teachers of Collegiate Mathematics. The Committee reported that the present supply of available teachers would not be adequate to meet the anticipated demands and urged all members of the Society to inform the Committee of persons who are available for appointment as teachers in the war training programs.

Titles and cross references to papers read at the meeting follow below. Papers 1 to 9 were read Friday morning, papers 10 to 18 Friday afternoon, papers 19 to 26 Saturday morning, papers 27 to 36 Saturday afternoon, and papers 37 to 56, whose abstract numbers are followed by the letter *t*, were read by title. Paper 6 was read by Professor Ulam, 11 by Dr. Caton, 16 by Dr. Wetzel, 25 by Professor Eilenberg, 26 by Dr. Erdös, and 27 by Mr. Fine. Mr. Kenneth Levenberg was introduced by Professor H. B. Curry, Dr. Angeline J. Brandt by Professor R. M. Thrall, and Mr. H. E. Salzer by Professor A. N. Lowan.

1. John DeCicco: *Dynamical and curvature trajectories in space.* (Abstract 49-11-297.)

2. Janet MacDonald: *Conjugate nets in asymptotic parameters.* (Abstract 50-1-46.)

3. Alice T. Schafer: *Two singularities of space curves.* (Abstract 49-11-300.)

4. Isaac Opatowski: *Isoperimetric problems in bending of cantilevers.* (Abstract 50-1-40.)

5. Kenneth Levenberg: *A method for the solution of certain non-linear problems in least squares.* (Abstract 49-11-286.)

6. L. W. Cohen and S. M. Ulam: *On the algebra of systems of vectors and some problems in kinematics.* (Abstract 50-1-38.)

7. G. Y. Rainich: *Noncommutative relations.* (Abstract 50-1-15.)

8. P. R. Halmos: *In general a measure preserving transformation is mixing.* (Abstract 50-1-44.)

9. J. L. Brenner: *The linear homogeneous group.* II. (Abstract 50-1-3.)

10. K. L. Nielsen: *On operators for linear partial differential equations.* (Abstract 50-1-30.)

11. W. B. Caton and Einar Hille: *On the class of functions* $H_p(1/2)$. (Abstract 50-1-23.)

12. S. E. Warschawski: *On Theodorsen's method of conformal mapping of nearly circular regions.* (Abstract 50-1-37.)

13. A. T. Lonseth: *The propagation of error in linear problems.* (Abstract 50-1-28.)

14. W. T. Reid: *Expansion theorems for boundary problems of the calculus of variations.* (Abstract 50-1-31.)

15. Anne L. Lewis: *Sufficiency proofs for the problem of Bolza in the calculus of variations.* (Abstract 50-1-27.)

16. H. S. Wall and Marion Wetzel: *Positive definite forms and convergence theorems for continued fractions.* (Abstract 49-9-216.)

17. W. J. Trjitzinsky: *Problems of representation and uniqueness for functions of a complex variable.* (Abstract 50-1-35.)

18. A. H. Copeland: *The nature of turbulence.* (Abstract 50-1-39.)

19. Dorothy M. Stone: *An algebraic characterization of measure algebras.* Preliminary report. (Abstract 49-11-249.)

20. J. V. Wehausen: *A remark concerning a set of completely continuous transformations.* (Abstract 49-11-274.)

21. J. W. T. Youngs: *On surfaces of class* K_1. Preliminary report. (Abstract 49-11-309.)

22. G. E. Schweigert: *Periodic analysis for onto-homeomorphisms of semi-locally-connected continua.* Preliminary report. (Abstract 50-1-51.)

23. M. E. Shanks: *Monotone decompositions of continua.* (Abstract 50-1-50.)

24. R. H. Fox: *The complete homotopy group.* (Abstract 49-11-306.)

25. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in groups.* (Abstract 50-1-9.)

26. Paul Erdős and S. M. Ulam: *Some combinatorial problems in set theory.* Preliminary report. (Abstract 50-1-25.)

27. N. J. Fine and Ivan Niven: *The probability that a determinant be congruent to a* (mod *m*). (Abstract 50-1-11.)

28. M. M. Day: *Arithmetic of ordered systems.* (Abstract 50-1-6.)

29. R. M. Thrall: *On the decomposition of modular tensors.* II. (Abstract 49-11-250.)

30. Ralph Hull: *A theorem on the unit groups of simple algebras.* (Abstract 50-1-14.)

31. A. A. Albert: *Quasiquaternion algebras.* (Abstract 50-1-2.)

32. L. R. Wilcox: *Modularity in Birkhoff lattices.* (Abstract 50-1-17.)

33. R. H. Bruck: *Some results in the theory of quasigroups.* (Abstract 49-11-244.)

34. Angeline J. Brandt: *The free Lie ring and Lie representations of the full linear group.* (Abstract 49-11-243.)

35. Wade Ellis: *Relations satisfied by linear operators on a vector space.* (Abstract 50-1-10.)

36. V. G. Grove: *The transformation of Čech.* (Abstract 50-1-45.)

37. Bernard Friedman: *A method of approximating the complex roots of polynomial equations.* (Abstract 49-11-280-*t*.)

38. D. G. Bourgin and C. W. Mendel: *Orthonormal sequences.* (Abstract 50-1-21-*t*.)

39. S. M. Ulam: *Theory of the operation of product of sets.* III. Preliminary report. (Abstract 50-1-36-*t*.)

40. Edward Kasner and John DeCicco: *Union-preserving transformations of space.* (Abstract 49-11-298-*t*.)

41. M. H. Heins: *On the problem of Milloux for functions analytic throughout the interior of the unit circle.* (Abstract 49-11-262-*t*.)

42. R. E. Johnson: *On the equation* $\chi\alpha=\gamma\chi+\beta$ *over an algebraic division ring.* (Abstract 49-11-246-*t*.)

43. M. E. Shanks: *The space of metrics on a compact metrizable space.* II. (Abstract 49-11-308-*t*.)

44. H. B. Curry: *The method of steepest descent for nonlinear minimization problems.* (Abstract 49-11-278-*t*.)

45. Dorothy M. Stone: *The representation of abstract measure functions.* (Abstract 49-11-273-*t*.)

46. R. H. Bruck: *Some results in the theory of linear non-associative algebras.* (Abstract 50-1-4-*t*.)

47. Morris Marden: *A recurrence formula for the solutions of certain partial differential equations.* (Abstract 50-1-29-*t*.)

48. Claude Chevalley: *Some properties of ideals in rings of power series.* (Abstract 50-1-5-*t*.)

49. H. E. Salzer: *New tables and facts involving sums of four tetrahedral numbers.* (Abstract 50-1-16-*t*.)

50. A. A. Albert: *Quasigroups.* II. (Abstract 50-1-1-*t*.)

51. L. R. Wilcox: *Theory of traffic light networks.* I. (Abstract 50-1-42-*t*.)

52. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* III. (Abstract 50-1-32-*t*.)

53. G. T. Whyburn: *Mapping classes for locally connected continua.* (Abstract 50-1-52-*t*.)

54. B. H. Colvin: *The expansion problem associated with a third order ordinary differential system of highly irregular type.* (Abstract 50-1-24-*t*.)

55. Alfred Brauer: *On certain limits.* (Abstract 50-1-22-*t*.)

56. R. L. Moore: *Concerning webs in the plane.* (Abstract 50-1-49-*t*.)

W. L. AYRES,
Associate Secretary

APPENDIX

Excerpts from Report of Treasurer

December 10, 1943

To the Board of Trustees of the
American Mathematical Society

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1943. The following remarks may be of interest.

Investment Portfolio

On November 30, 1943 the market value of the securities held for Invested Funds was $6,208 less than book value. This is to be compared with the total amount, $17,416, held in accounts "Reserve for Investment Losses" and "Profit on Sales of Securities." This total amount of reserves has now reached nearly ten per cent of the total value of Invested Funds. In view of the conservative character of many of the securities, these reserves may be considered sufficient protection against contingent depreciation in market value. The considerable increase, $8,843, in this profit account during the year reflects two policies embodied in the recommendations of the Custodian and followed by the Board of Trustees—sales of railroad bonds which had been held on the books at depreciated values, and sales of certain common stocks when rising market values increased the percentage of common stocks in the portfolio beyond the point deemed desirable.

The market value of securities held for Current Funds was $654 less than the book value. Profit on sales of securities for Current Funds amounting to $612 is being carried as a reserve against losses to these funds.

The investment portfolio now includes government bonds 29%, other bonds 24%, preferred stocks 11%, common stocks 28%, cash in savings banks 8%.

Income from Investments

Income received during the year from investment of Current Funds amounted to $1,254, exclusive of income earmarked for International Congress, $96, and Reilly Fund, $22. This represents a return of slightly under $2\frac{3}{4}\%$, computed on average book value of investments.

Income on Invested Funds amounted to $6,352, representing a return of 3¼% computed similarly. Total investment income from all sources was thus $7,724, representing a return of nearly 3½%. In comparing this income and yield with $8,794 and approximately 4% for the fiscal year 1942, there is reflected to some extent the changes in the character of the securities resulting from the sales mentioned above and the purchase of substantial amounts of government bonds. Also the figures for 1942 were somewhat too high to be truly representative, because of the receipt of a considerable sum of accrued back interest on certain railroad bonds then held. In view of the decreased rate of return, it is particularly fortunate that the Henderson Estate provided additional income of $4,438.

Increase in Surplus

The increase in Surplus of $9,553, as compared with $5,153 in the preceding fiscal year requires some explanation. If it be noted that 1942 income from the Henderson Estate amounting to $2,275 had not been added to Surplus on November 30, 1942, but was so added during the current fiscal year, more proper figures for comparison are seen to be $7,278 for 1943 and $7,428 for 1942. That the net result of the year's operations proved about as favorable as that for the preceding year is particularly satisfactory in view of the founding of Mathematical Surveys, made possible partly by the use of income from special funds (in the case of the Reilly Fund to such an extent that there is a small deficit in income account!), of somewhat increased expenses for other purposes, of the support of the War Policy Committee, and of the decrease in income from the Society's investments. However, a grant of $2,500 from the Rockefeller Foundation, half of which has been received, prevented the support of the War Policy Committee from seriously depleting Surplus. And an increase in membership that resulted in an increase in dues, together with a slight increase in subscriptions to the Bulletin and to the Transactions, were other favorable factors that enabled the Society to come successfully through a year which at its beginning threatened to be rather difficult financially.

Respectfully submitted,

BENNINGTON P. GILL,
Treasurer

BALANCE SHEET

Assets

	November 30, 1943	November 30, 1942
CURRENT FUNDS:		
Cash	$ 19,846.11	$ 19,177.78
Investments	36,731.06	33,470.47
	$ 56,577.17	$ 52,648.25
INVESTED FUNDS:		
Cash	$ 1,273.67	$ 1,409.62
Investments	176,721.29	163,195.68
	$177,994.96	$164,605.30
TOTAL ASSETS	$234,572.13	$217,253.55

Liabilities

CURRENT FUNDS:		
Mathematical Reviews	$ 14,252.62	$ 16,184.69
Colloquium	5,761.69	2,805.04
Mathematical Surveys	840.03	
Reprinting Funds	2,239.10	1,329.15
Prize Funds and Other Special Funds Accumulated Income	2,296.18	4,848.52
Reilly Fund Principal		3,968.13
International Congress	6,146.06	6,050.33
Miscellaneous	80.99	173.99
Income from Henderson Estate		2,275.00
Sinking Fund	592.34	515.99
Profit on Sales of Securities	612.86	295.73
Surplus	23,755.30	14,201.68
	$ 56,577.17	$ 52,648.25
INVESTED FUNDS:		
Endowment Fund Principal	$ 60,110.68	$ 60,110.68
Prize Funds and Other Special Funds	31,033.22	26,935.07
Life Membership and Subscription Reserve	4,434.81	4,835.09
Colloquium	5,000.00	5,000.00
Mathematical Reviews	60,000.00	60,000.00
Reserve for Investment Losses	4,385.89	3,537.46
Profit on Sales of Securities	13,030.36	4,187.00
	$177,994.96	$164,605.30
TOTAL LIABILITIES	$234,572.13	$217,253.55

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1942–1943

	1943		1942	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS:				
Dues—Ordinary Memberships	$ 16,514.71		$ 15,681.19	
Dues—Contributing Memberships	859.63		1,054.00	
Dues—Institutional Memberships	6,616.52		7,347.35	
Initiation Fees	378.84		665.00	
Investment Income	8,612.83		4,643.14	
Miscellaneous	39.96		34.73	
Grant from Rockefeller Foundation for War Policy Committee	1,250.00			
GENERAL DISBURSEMENTS:				
Secretaries		$ 5,590.02		$ 5,136.86
Treasurer		1,093.58		794.36
Officers' Traveling		581.23		643.61
Library		893.15		569.95
Committee Expenses		1,584.17		67.92
Emergency		4.90		5.71
Total	$ 34,272.49	$ 9,747.05	$ 29,425.41	$ 7,218.41
Excess of General Receipts	$ 24,525.44		$ 22,207.00	
PUBLICATION:				
Bulletin	$ 1,710.80	$11,536.88	$ 1,673.13	$12,101.37
Bulletin Reprinting	626.71		467.30	
Transactions	5,886.83	8,309.42	5,787.17	9,373.59
Transactions Reprinting	1,172.40	889.16	274.58	201.59
Colloquium	3,951.38	994.73	3,814.97	6,197.07
Mathematical Reviews	10,235.88	12,167.95	10,809.17	11,548.93
Mathematical Surveys	3,600.89	2,760.86		
Semicentennial Publications	76.35		95.90	
American Journal		2,500.00		2,500.00
Total	$ 27,261.24	$39,159.00	$ 22,922.22	$41,922.55
Excess of Cost of Publication		$11,897.76		$19,000.33
MISCELLANEOUS:				
Special Funds (including Congress)	$ 1,273.41	$ 3,600.00	$ 1,442.39	$ 2,100.00
Profit on Sales of Securities	9,160.49		2,713.28	
Income from Henderson Estate			2,275.00	
Gift from R. C. Archibald		80.97	100.00	
Credits not Allocated		12.03	73.99	103.86
From Surplus:				
Transactions Reprinting		750.00		100.00
Mathematical Reviews		300.00		
Mathematical Surveys		1,000.00		
Total	$ 10,433.90	$ 5,743.00	$ 6,604.66	$ 2,303.86
Difference	$ 4,690.90		$ 4,300.80	
Net Change in Assets	$ 17,318.58		$ 7,507.47	
ASSETS BEGINNING OF YEAR	$217,253.55		$209,746.08	
ASSETS END OF YEAR	$234,572.13		$217,253.55	

EDWARD BURR VAN VLECK—IN MEMORIAM

The death of Edward Burr Van Vleck on June 2, 1943, at almost eighty years of age, in Madison, Wisconsin, will bring regret to a wide circle of friends and to American mathematicians generally. His creative mathematical work, his important role in the development of American mathematics since 1890, and his qualities of personality were such as to make all feel that something of inestimable and characteristic value has departed and yet remains in treasured remembrance.

Van Vleck was born in Middletown, Connecticut, on June 7, 1863, of Knickerbocker stock, for the first Van Vleeck came from Amsterdam, Holland, to New Amsterdam (New York) in 1658. His father was a distinguished American mathematical astronomer, Professor John Monroe Van Vleck, in whose memory has been erected the Van Vleck Observatory at Wesleyan University, given by a brother. The scientific tradition, thus doubly established in the Van Vleck family, has been notably continued to a third generation by his son, Professor John Hasbrouck Van Vleck, now Professor of Mathematical Physics in Harvard University.

The general account of Edward Burr Van Vleck's life is simple. As a boy, college student, and instructor, he developed in the typical New England environment of Middletown, living at home with his parents and three sisters (one of whom survives him) until about twenty-five years of age, obtaining an A.B. degree from Wesleyan University at twenty-one years of age and an A.M. degree three years later. Then he spent two years as a graduate student at Johns Hopkins University, where he veered from the field of physics to that of mathematics. After this came three further years at the University of Göttingen, Germany, where he was inspired by the great mathematician Felix Klein. He obtained his Ph.D. degree in 1893 at Göttingen at about thirty years of age. Thus he entered upon his mathematical research work somewhat later than usual, a fact which he always felt to be regrettable.

His first college post was that of instructor for two years at the University of Wisconsin, after which he taught for ten years at his alma mater, Wesleyan University, becoming associate and full professor there. In 1906 he went to the University of Wisconsin as Chairman of the Department of Mathematics, and, except for a half-year (1919–20) when he was visiting lecturer in the Department of Mathematics at Harvard University, and for various periods of travel in

this country and abroad, remained in Madison thereafter. He became professor emeritus in 1929.

Behind this academic outline there was always a full and gracious life. Van Vleck married Miss Hester Laurence Raymond of North Lyme, Connecticut, shortly after he returned from his studies in Germany. In both Middletown and Madison, the Van Vlecks added much to the college and community life. Their lovely home in Madison, overlooking Lake Mendota, will be remembered by many as a center from which radiated kindness, friendship, and delightful hospitality to colleagues and their families, to students, and indeed to all those about them.

One may conjecture that Van Vleck was a very thoughtful and rather shy boy. Throughout his life he disliked all disputes, so that, even to his own disadvantage, he would avoid controversies with others. He liked urbane living and achieved it to a notable extent. However, when questions of principle arose, he stood his ground firmly and said precisely what he thought, although in polite form.

To those who knew him during his maturer years, he appeared as a clearsighted and liberal-minded idealist never deviating from the highest standards for himself, and yet fully appreciating the qualities of others who had a different outlook. His life was built around the motives of friendship and genuine scientific interest to a greater degree than is usually the case in the American academic scene.

Van Vleck's principal outside interests were reading and travel, both of which he shared with Mrs. Van Vleck. During the last half of his life he collected Japanese prints, and grew to be a connoisseur in this field.

On the mathematical side, Van Vleck owed much, for stimulus and inspiration, to Felix Klein. His thesis treated the development in continued fractions of integrals, forming solutions of certain linear differential equations of the second order, such as Lamé's integral. These integrals fall into natural families, any three "contiguous" members being connected by homogeneous linear difference equations. These integrals had appeared earlier in classical developments in continued fractions due to Gauss and others. This was a field which lay at the very heart of that part of analysis in which Klein was most interested. In his thesis Van Vleck extended considerably the types of such developments in a manner somewhat analogous to that devised by Pade in the similar development of arbitrary analytic functions. His thesis showed that not only was the young Van Vleck a master of the field involved but that he possessed definite creative power.

The question of the convergence of these continued fractions was

left untouched in the thesis. For continued fractions of special form, this was taken up by him in three papers published in the second and fourth volumes of the Transactions of the American Mathematical Society (1901, 1904). A well known and general theorem there proved is that if in the continued fraction

$$\frac{1}{\alpha_1 + i\beta_1} + \frac{1}{\alpha_2 + i\beta_2} + \cdots \qquad i = (-1)^{1/2}$$

the real quantities α_i have the same sign while the real quantities β_i are alternately positive and negative, then the continued fraction will converge if the series $\sum(\alpha_u + i\beta_u)$ converges.

In his Colloquium Lectures of 1903, *Divergent series and continued fractions*, he first reviews the recent work of Poincaré, Stieltjes and others on divergent series (Part I) and then passes on to consider continued fractions (Part II). The two subjects are intertwined because "Both historically and prospectively one of the most suggestive and important methods of investigating power series is by the instrumentality of algebraic continued fractions." Van Vleck shows clearly how, from the broad point of view, the theory of continued fractions and their generalizations is closely related to the theory of linear homogeneous difference equations whose coefficients are polynomials in a complex variable. He wrote only one paper directly concerned with linear difference equations (Trans. Amer. Math. Soc., 1912) in which he extended a classic theorem of Poincaré. Doubtless the reason for this limited publication was the rapidity with which appeared new developments in the field by Nörlund and Galbrun abroad and by Carmichael, myself, and others in this country. However, while instructor at Wisconsin I had attended a graduate course given by Van Vleck in which he discussed various open problems concerning linear difference equations in a suggestive and stimulating way; while Carmichael shortly thereafter was a graduate student with me at Princeton. One must therefore look upon Van Vleck as an essential factor in the American contributions to linear homogeneous difference equations.

I have no doubt that when Van Vleck saw this promising field in which he had begun to work taken up so quickly by others, there was no tinge of regret on his part, for he would know that there were many other beautiful fields to be explored, and would feel that it was relatively unimportant from the larger point of view what the formal assignment of credit might be, so long as the subject itself was developed. It was enough for him that all should work together earnestly and sincerely for the increase of mathematical knowledge.

For a while Van Vleck seriously contemplated writing a book on functional equations, but the apparent lack of mathematical unity in the field prevented him from doing so. He wrote interesting papers dealing with special functional equations, in particular those satisfied by the elliptic ϑ-functions (Trans. Amer. Math. Soc., 1916); this last was the joint work of himself and one of his most gifted students, F. T. H'Doubler.

The range covered by Van Vleck in his more than thirty published articles was wide. He had a very keen analytic mind well adapted to the study of the niceties of point set theory, as manifested for instance in his article on non-measurable sets (Trans. Amer. Math. Soc., 1916). He also occupied himself with questions concerning the roots of algebraic equations and of the hypergeometric series, with special functions defined by differential equations, with the combination of substitutions, linear substitutions, and so on. In my opinion he resembled Bôcher in the respect that as the years went on, the self-imposed demand for elegance and simplicity exercised an inhibitive influence upon his production.

His general articles, *The influence of Fourier's series upon the development of mathematics* (Science, 1912), *The role of the point-set theory in geometry and dynamics* (Presidential Address, Bull. Amer. Math. Soc., 1915), *Current tendencies of mathematical research* (ibid., 1916), and *On the location of roots of polynomials and entire functions* (ibid., 1928) were thought-provoking essays and showed not only a wide knowledge and balanced perspective but an unusual ability to find the important open problems in the fields concerned. For example, in the second of these articles he states that Poincaré's definition of (dynamical) probability is equivalent to one in terms of the Lebesgue integral, although he does not give his proof. Carathéodory's well known proof of this equivalence appeared only in 1919.

In the teaching of undergraduates and of graduate students Van Vleck was clear and stimulating. At Wisconsin he gave himself unsparingly to the direction of the thesis work of undergraduates specializing in mathematics. As Chairman of the Department of Mathematics at Wisconsin for many years, he was deeply considerate of the members of the staff, completely democratic in spirit, and absolutely impartial. The friendship between him and his colleague Professor Ernest Skinner was one of great depth on both sides.

The mathematical reputation of Van Vleck was an international one. His relations with French mathematics were particularly close and sympathetic. Moreover his affection and admiration for France were profound. He was decorated "Officier de l'instruction publique"

in 1920 in recognition of his services as teacher and investigator and for his work during the First World War. The University of Groningen on the three hundredth anniversary of its founding made him an Honorary Doctor in Mathematics and Physics in 1914. He, together with E. H. Moore and W. F. Osgood, received an honorary LL.D. at Clark University in 1909. At the twenty-fifth anniversary celebration of the University of Chicago in 1916, he received the honorary degree of Doctor of Science. His election to membership to the National Academy of Sciences came in 1911.

Van Vleck was President of the American Mathematical Society for the period 1913–1915. In earlier days he had been an effective and conscientious Editor of the Transactions. E. H. Moore, Maxime Bôcher, and H. S. White were others among his near contemporaries who were similarly honored by election to both of these important offices. All of these men served American mathematics faithfully and well. It will be remembered by many that, at one moment when the unity of the American Mathematical Society was somewhat in jeopardy, Van Vleck exerted an especially constructive influence.

It would have been interesting to have had Van Vleck himself formulate his outlook upon the world. Of his own volition he would of course never have done so. But, had he been somehow persuaded to give his philosophy of living, it would have expressed a large measure of tolerance towards others. As a young colleague of his at Wisconsin, I recall asking him about a great writer whose personal life had been far from admirable. Van Vleck replied very quickly: "Don't you think that what he has done for the world of literature more than made up for any personal defects?" Above all, he would have emphasized the virtues of kindness and friendship, of disinterested devotion to intellectual and other ideals, and of a due regard for the amenities of our civilization.

GEORGE D. BIRKHOFF

BOOK REVIEWS

Differentialgeometrien in den Kugelräumen. Vol. II. *Laguerresche Differentialkugelgeometrie.* By T. Takasu. Tokyo, Maruzen, 1939. 20+444 pp.

This volume is the second of a series of three, in which Professor Takasu attempts to give a unified and systematical representation of his work in the differential geometry of sphere space. Concentrating on the geometries of Möbius, Laguerre, and Lie, he presents in this volume his results in Laguerre geometry. In his exposition he follows as closely as possible the usual pattern of euclidean differential geometry, to the systematic study of which the author has contributed in several previous papers.

Following this scheme of development of his subject, Professor Takasu has divided his field into five parts. The first chapter explains the fundamental elements and the coordinates of Laguerre geometry, and shows how we can obtain a correspondence between euclidean three space and a Laguerre plane by means of minimal (isotropic) projection. The group of congruent transformations in space corresponds, in this transformation, to the Laguerre group of the plane. Then follows, in Chapter II, the Laguerre generalization of plane euclidean theory of curves, which is the theory of Laguerre invariants of oriented circles. In Chapter III we find the analogous theory of oriented spheres, corresponding to space curves in ordinary geometry. The Laguerre analogues of surfaces are congruences of spheres, which are discussed in Chapter IV. The final chapter deals with systems of cones, which correspond to line congruences in ordinary space.

This gives us quite a good picture of the structure and of the content of Laguerre differential geometry. The author has placed all previous theories, as those of Darboux, of Ribaucour, of Study, in their proper place and has enriched his work with the investigations of Blaschke and his school. He has shown how much more there is to Laguerre geometry than we could have expected from the study of even the richest other author, and how his methods allow the systematic penetration of all fields into which ordinary differential geometry has thrown its light. We mention, as examples out of a great many other results, the author's treatment of "L-developables" ("L-Torsen"), of L-minimal surfaces, and of Bonnet's plane coordinates. He also ventures into differential geometry in the large, as in a Laguerre version of the four vertex theorem.

We believe that Professor Takasu would have increased the useful-

ness of his work, if he had dwelled longer on the elementary material of his first chapter. The reader who tackles his book without previous knowledge of Laguerre geometry will find the going hard at the beginning. Once he has overcome the initial difficulties, however, he will be richly rewarded by the great number of beautiful results which fall into his lap, and by the mastery of a method which will allow him to find many more results by his own effort.

The book has a complete bibliography and an excellent index.

D. J. STRUIK

The theory and applications of harmonic integrals. By W. V. D. Hodge. Cambridge, University Press; New York, Macmillan, 1941. 9+281 pp. $4.50.

This is one of those books which everyone who specializes in a particular branch of group theory, of the theory of algebraic surfaces, of the theory of Riemann surfaces, of topology or of the tensor analysis should consult. It shows how all these different fields are connected, and not connected in some superficial way or in the form of an analogy, but in an essential manner, so that interesting and profound theorems in one field cannot be understood without a thorough knowledge of other fields. In reading this book one is reminded of books like Klein's "Ikosaeder," which is also a blend of several important fields. The task of the reviewer of such a book is hard, because he has seldom the enviable mastery of the different branches of mathematics which the author possesses. At the same time he must praise the author for the beautiful exposition of so many and different fields.

There are chapters on Riemannian manifolds, on integrals and their periods, on harmonic integrals, on their applications to algebraic varieties and on their applications to the theory of continuous groups. The first chapter, on Riemannian manifolds, is divided into a part on tensor calculus and into a part on the topology of such manifolds. All these topics are prepared in a very careful way, with precautions which will satisfy strict demands of rigor. There are references at the end of each chapter.

To indicate the contents of this book by simply copying the chapter headings is to do a great injustice to the author and his work. There is a leading thought in the choice of subject matter, and that is the study of harmonic integrals. Harmonic integrals, however, are not the capricious invention of an imaginative scholar. They appear quite naturally in the generalization of the problems set by

Riemann's theory of integration on an algebraic surface, and in particular if we ask for the existence of m-fold integrals on an algebraic manifold of m dimensions which are everywhere finite and of zero period. The investigation is facilitated by the Cartan notation in which differentials alternate and in which the exterior derivative of a "p-form" $A_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p} = p!\, A$ is defined in such a way that Stokes' theorem takes the form $\int A = \int A_z$, the second integral taken over a chain C_p and the first one over its boundary. This again requires not only some understanding of tensor algebra, but also of the topology of manifolds, which the author provides in his early chapters. There are two theorems of DeRham (J. Math. Pures Appl. (9) vol. 10 (1931) p. 115) which lead up to the heart of the problem and which are demonstrated in this book. The first of these theorems states (p. 88) that if Γ_p^i $(i = 1, \cdots, R_p)$ is any base for the p-cycles of a manifold M, then there exists a regular closed ϕ-form on M for which the periods $\nu^i = \int \phi$ (on Γ_p^i) are R_p arbitrary real numbers. The second theorem of DeRham states (p. 100) that if ϕ is a closed form on M whose integral has all its periods equal to zero, then ϕ is a null form. Closed forms are forms with exterior derivatives $A_z = 0$, and null forms are the exterior derivatives of other forms, hence forms with zero periods.

The author derives DeRham's second theorem from his main existence theorem for harmonic integrals. This requires the introduction of a Riemannian metric. A function which satisfies the Laplacian equation in this metric is called harmonic. A p-form P is harmonic when it is regular and closed, as well as its dual, the $(n-p)$-form P^*; $p!P$ and $p!P^*$ correspond to the harmonic tensors $P_{i_1 \cdots i_p}$ and $g^{1/2} P^{k_1 \cdots k_p}_{\cdots\cdots\cdots k_1 \cdots k_p k_{p+1} \cdots k_n}$. A harmonic integral is the integral of a harmonic form. The gradient of a harmonic function ϕ_i is a simple example of a harmonic tensor. The real and imaginary paths of an integral of a complex variable are harmonic.

One of the main objects of the third chapter is the proof that on a Riemannian manifold M such harmonic integrals actually exist. The existence theorem states (p. 119) that there exists a harmonic integral having arbitrarily assigned periods on R_p independent p-cycles of M. The proof requires considerable labor and demands the knowledge of the theory of integral equations. Once established, the main existence theorem is used for the study of algebraic varieties. It is interesting that an ancient theorem of G. Mannoury (Nieuw Archief voor Wiskunde vol. 4 (1898) p. 112)—the deeper meaning of which had always puzzled the reviewer—is used to show how Riemannian manifolds can be constructed corresponding to an algebraic variety V_m

without singularities in a projective space S_r. In this representation the S_r is given by a real analytical locus of $2r$ dimensions. A large number of applications are presented together with a classification of harmonic integrals. This again connects with Lefschetz' classification of p-cycles. Period matrices of so-called "effective" integrals are studied in detail, "effective" integrals being integrals of p-forms for which a certain related $(p-2)$-form is a null form. Among the many results are several due to Severi and others which are new. In the application to continuous groups the representation of closed semi-simple groups by group manifolds is used. The main purpose of this chapter is to show how harmonic integrals provide a convenient method of discussing the invariant integrals of Cartan's theory.

A flaw in the proof of the theorem establishing the existence of harmonic integrals with preassigned periods has been corrected by H. Weyl in the Annals of Mathematics vol. 44 (1943) pp. 1–6.

The nineteen pages devoted to the exposition of the tensor calculus can be recommended to those who like a short and precise introduction into this theory.

D. J. STRUIK

NOTES

Beginning with this issue the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY will be copyrighted. It is not the desire of the editors to restrict in any way the use of mathematical material from the BULLETIN nor to deny the right to quote from its text. However, it is desired to regulate the republication or sale of papers that have been published in the BULLETIN.

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, Leon Alaoglu, A. A. Albert, G. E. Albert, Reinhold Baer, M. A. Basoco, E. F. Beckenbach, E. T. Bell, Harry Blumberg, Herbert Busemann, R. V. Churchill, George Comenetz, H. S. M. Coxeter, M. M. Day, R. H. Fox, K. O. Friedrichs, Guido Fubini, D. W. Hall, Selig Hecht, M. H. Heins, M. R. Hestenes, T. H. Hildebrandt, Ralph Hull, Nathan Jacobson, R. L. Jeffery, Irving Kaplansky, A. J. Kempner, R. B. Kershner, E. P. Lane, R. E. Langer, Walter Leighton, D. C. Lewis, E. R. Lorch, Szolem Mandelbrojt, G. M. Merriman, W. E. Milne, F. J. Murray, Oystein Ore, G. H. Peebles, F. W. Perkins, E. L. Post, G. B. Price, Hans Rademacher, Tibor Radó, H. W. Raudenbush, W. T. Reid, Arthur Rosenthal, A. C. Schaeffer, Peter Scherk, I. J. Schoenberg, Wladimir Seidel, F. C. Smith, N. E. Steenrod, A. H. Stone, Otto Szász, Alfred Tarski, E. W. Titt, J. W. Tukey, S. M. Ulam, R. J. Walker, H. S. Wall, Morgan Ward, Anna P. Wheeler, W. M. Whyburn, L. R. Wilcox, Aurel Wintner, Leo Zippin, and Antoı Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 53 and 54: Emil Artin, Reinhold Baer, E. G. Begle, Garrett Birkhoff, R. P. Boas, R. H. Bruck, E. H. Buchanan, Herbert Busemann, Alonzo Church, R. P. Dilworth, G. A. Hedlund, R. L. Jeffery, N. H. McCoy, E. J. McShane, F. J. Murray, George Pólya, C. E. Rickart, R. M. Robinson, O. F. G. Schilling, D. C. Spencer, E. R. Stabler, Otto Szász, J. D. Tamarkin, D. V. Widder, and Antoni Zygmund.

Professor Philip Franklin of Massachusetts Institute of Technology has received an alumni medal from the College of the City of New York.

Mr. J. Lifshitz of the National University of Mexico has been awarded a Guggenheim fellowship to study at Harvard University under the direction of Professor G. D. Birkhoff.

Professor F. G. Reynolds of the College of the City of New York has been given the title of professor emeritus.

Professor Shiing-shen Chern, Dr. Kurt Gödel, Professor Wolfgang Pauli, and Professor C. L. Siegel are in residence at the Institute for Advanced Study for this academic year.

Dr. Claire F. Adler has been appointed to an assistant professorship at New York University.

Dr. G. E. Albert of Ohio State University has been promoted to an assistant professorship

Dr. Warren Ambrose of Princeton University has been appointed to an assistant professorship at the University of Michigan.

Dr. Valentin Bargmann has been appointed an assistant at the Institute for Advanced Study.

Assistant Professor P. O. Bell of the University of Kansas has been promoted to an associate professorship.

Assistant Professor D. G. Bourgin of the University of Illinois has been promoted to an associate professorship.

Assistant Professor J. D. Burk of the University of Toronto has been promoted to an associate professorship.

Assistant Professor C. L. Buxton of Clarkson College of Technology, Potsdam, New York, has been promoted to an associate professorship of physics.

Associate Professor Ethel B. Callahan of Shepherd State Teachers College, Shepherdstown, West Virginia, has been appointed to an associate professorship at Hartwick College, Oneonta, New York.

Assistant Professor R. H. Cameron of Massachusetts Institute of Technology has been promoted to an associate professorship.

Associate Professor E. J. Camp of Macalester College, St. Paul, Minnesota, has been promoted to a professorship.

Dr. Josephine H. Chanler of the University of Illinois has been promoted to an assistant professorship.

Dr. R. F. Clippinger of Carnegie Institute of Technology has been promoted to an assistant professorship.

Mr. A. J. Coleman has been appointed lecturer in mathematics at Queen's University in Kingston, Ontario, Canada.

Assistant Professor N. B. Conkwright of the University of Iowa has been promoted to an associate professorship.

Mr. W. S. H. Crawford of the University of Minnesota has been appointed to an assistant professorship at Mount Allison University.

Sister James S. Creane has been appointed to an assistant professorship at Fontbonne College, St. Louis, Missouri.

Associate Professor Wayne Dancer of the University of Toledo has

been promoted to a professorship.

Associate Professor P. H. Daus of the University of California at Los Angeles has been promoted to a professorship.

Mr. U. P. Davis of the University of Florida has been promoted to an assistant professorship.

Professor R. C. Dragoo of Southwestern Institute of Technology has been appointed to an assistant professorship at the University of Oklahoma.

Assistant Professor J. L. Doob of the University of Illinois has been promoted to an associate professorship.

Associate Professor W. L. Duren of Tulane University has been promoted to a professorship.

Mr. R. M. Foster of Bell Telephone Laboratories has been appointed professor and head of the department of mathematics of the Polytechnic Institute of Brooklyn.

Assistant Professor A. H. Fox of Union College has been promoted to an associate professorship.

Dr. L. M. Garrison of Alabama Polytechnic Institute has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Assistant Professor W. O. Gordon of Pennsylvania State College has been promoted to an associate professorship.

Mr. J. B. Greeley of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at Lafayette College.

Mr. H. L. Harter of the University of Illinois has been appointed to a professorship of physics at Missouri Valley College, Marshall, Missouri.

Dr. A. D. Hestenes of Carnegie Institute of Technology has been promoted to an assistant professorship.

Assistant Professor E. H. C. Hildebrandt of the State Teachers College, Upper Montclair, New Jersey, has been appointed to an assistant professorship at Northwestern University.

Assistant Professor Evan Johnson of Pennsylvania State College has been promoted to an associate professorship.

Assistant Professor P. W. Ketchum of the University of Illinois has been promoted to an associate professorship.

Assistant Professor L. A. Knowler of the University of Iowa has been promoted to an associate professorship.

Mr. Max LeLeiko has been appointed lecturer in mathematics at the University of North Carolina.

Dr. A. T. Lonseth of Iowa State College has been promoted to an assistant professorship.

Assistant Professor C. T. Male of Union College has been promoted to an associate professorship.

Dr. H. J. Miles of the University of Illinois has been promoted to an assistant professorship.

Associate Professor F. H. Miller of Cooper Union School of Engineering has been promoted to a professorship and has been appointed head of the department of mathematics.

Mr. H. G. Means has been appointed to an associate professorship at Lehigh University.

Assistant Professor G. E. Moore of the University of Illinois has been promoted to an associate professorship.

Assistant Professor S. B. Myers of the University of Michigan has been promoted to an associate professorship.

Mr. J. D. Novak of the University of Minnesota has been appointed head of the department of mathematics and physics at MacMurray College, Jacksonville, Illinois.

Dr. W. H. Pell and Dr. Vladimir Morkovin of Brown University have accepted positions as research engineers in aerodynamic design at the Bell Aircraft Corporation, Buffalo, New York.

Dr. H. V. Price of the State University of Iowa has been promoted to an assistant professorship.

Associate Professor W. C. Randels of the University of Oklahoma has resigned to serve with the North American Aviation Company in Los Angeles.

Professor H. W. Reddick of Cooper Union has been appointed adjunct professor at New York University.

Assistant Professor P. K. Rees of Southern Methodist University has been appointed to a professorship at Southwestern Louisiana Institute.

Assistant Professor G. de B. Robinson of the University of Toronto has been promoted to an associate professorship.

Assistant Professor W. J. Robinson of Marshall College, Huntington, West Virginia, has been appointed to an assistant professorship at Allegheny College.

Dr. D. H. Rock of Brown University has been appointed to an assistant professorship at Rhode Island State College.

Assistant Professor W. O. Rogers of Pennsylvania State College has been promoted to an associate professorship.

Assistant Professor George Sauté of Cleveland College has been appointed to an associate professorship at Rollins College.

Assistant Professor A. C. Schaeffer of Stanford University has been promoted to an associate professorship.

Dr. H. M. Schwartz of the University of Illinois has been appointed to an assistant professorship at the University of Idaho.

Dr. G. E. Schweigert of the University of Missouri has been appointed to an assistant professorship at Purdue University.

Dr. R. E. Street has been appointed to a visiting lectureship in physics at Dartmouth College.

Assistant Professor H. P. Thielman of Iowa State College has been promoted to an associate professorship.

Professor C. C. Wagner of Pennsylvania State College has been named assistant dean of the School of Liberal Arts.

Dr. George Whaples of Indiana University has been appointed to an assistant professorship at the University of Pennsylvania.

Associate Professor J. K. Whittemore of Yale University has retired.

Associate Professor James Edgar Davis of Central Y.M.C.A. College, Chicago, Illinois, has been appointed an associate at the University of Illinois.

Drs. O. K. Bower, Leonard Bristow, M. M. Day, and J. W. Peters of the University of Illinois have been promoted from the rank of instructor to that of associate.

The following appointments to instructorships are announced: Albright College, Reading, Pennsylvania: Mr. C. C. Miesse; University of Arizona: Dr. O. B. Ader; Barnard College, Columbia University: Miss Louise M. Comer; Brown University: Mr. W. E. Barnes, Mr. W. M. Kincaid, Mr. P. T. Mielke; University of Chicago: Dr. Anne L. Lewis, Dr. Janet MacDonald; Cooper Union: Mr. S. G. Roth; Duke University: Dr. L. I. Wade; Hunter College: Miss Louise W. Miller; University of Illinois: Mr. W. F. Atchison, Dr. Theodore Bedrick, Miss Janie C. Lapsley; Massachusetts Institute of Technology: Mr. W. S. Loud; University of Michigan: Mr. B. J. Lockhart, Dr. C. A. Truesdell; University of Minnesota: Mr. H. D. Colson, Dr. C. M. Jensen; University of Nebraska: Mr. H. W. Linscheid; Pennsylvania State College: Dr. A. B. Cunningham; Princeton University: Mr. Richard Bellman; Purdue University: Dr. Fred Kiokemeister; University of Rochester: Dr. Dorothy L. Bernstein; State College of Washington: Dr. P. F. Neményi; Yale University: Mr. A. S. Day, Mr. R. E. Fullerton, Mr. J. S. Leech.

Dr. G. T. Bennett of Emmanuel College, Cambridge, England, died October 11, 1943, at the age of seventy-five years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

1. A. A. Albert: *Quasigroups.* II.

A loop is a quasigroup with an identity element. In this part it is shown that H is a normal divisor of a loop G if and only if $(xH)(yH) \subset (xy)H$, $xH \subset (xH)h$, $(xy)H \subset x(yH)$ for every x and y of G and h of H. The intersection and union of two normal divisors of G are normal divisors of G, and the standard theorems used to prove the Jordan-Hölder theorem and the Schreier refinement theorem are valid for loops (although the proofs are very different). The paper shows how to extend the notion of solvable group to loops and also proves that various results of the theory of groups are also valid for loops. A construction is given of all loops with a given normal divisor and a given quotient loop, and the theory is applied to give an explicit determination of all loops of order six with a subloop of order three. Finally it is shown that all quasigroups of order five not isotopic to the group are isotopic to each other. (Received October 15, 1943.)

2. A. A. Albert: *Quasiquaternion algebras*

A quasiquaternion algebra has a basis $1, i, j, ji$ over a field F such that $i^2 = a + bi$, $ij = j(1 - i)$, $j^2 = c$ for a, b, c in F, $c \neq 0$, $b \neq 1$. Also $b = 0$ when F has characteristic different from two and one writes $A = (a, c)$ in this case. All quadratic subalgebras are determined and it is shown that (a, c) is isomorphic to (a_0, c_0) if and only if $a_0 = a$, $c_0 = d^2c$ for $d \neq 0$ in F. The results in the characteristic two case are slightly different. A quasiquaternion algebra is a division algebra if and only if the algebras $F[i]$ and $F[j]$ are fields. If F is a finite field of q elements there are division algebras of this kind over F if and only if q is odd. Then there are $1/2(q-1)$ such algebras not isomorphic in pairs. The question as to when two algebras over F of characteristic not two are isotopic is completely solved for quasiquaternion division algebras. (Received October 15, 1943.)

3. J. L. Brenner: *The linear homogeneous group.* II.

Let (x_i) represent an n-tuple (vector) whose elements are residue classes mod p^r (p, prime; r, positive integer). The p^{nr} vectors (x_i) form a group $\mathfrak{A}$ under the operation vector addition: $(x_i) + (y_i) = (z_i)$, $z_i \equiv x_i + y_i \pmod{p^r}$. An automorphism of $\mathfrak{A}$ may be defined by specifying the n images (a_{ji}) of the generators $e_j = (0, \cdots, 1_j, \cdots, 0)$, where $\det(a_{ji}) \not\equiv 1 \pmod{p}$. $\mathfrak{G}_{p,n,r}$ is the group of these automorphisms. In this article the lattices of normal and of characteristic subgroups of $\mathfrak{G}$ are described; the lattices are distributive except when $n = p = 2$, in which case they are not distributive. $\mathfrak{N}_e$ con-

sists of all matrices in $\mathfrak{G}$ which fix every element of order p^e in $\mathfrak{A}$. $\mathfrak{R}_e$ is characteristic in $\mathfrak{G}$. This article will appear in Ann. of Math. vol. 45 (January, 1944). (Received November 1, 1943.)

4. R. H. Bruck: *Some results in the theory of linear non-associative algebras.*

An algebra is defined to be isotopically simple if every isotope is simple. The theory of linear algebras is shown to be reducible to that of isotopically simple algebras. The following exhaustive classification of algebras is convenient: (I) all algebras with unit elements, and their isotopes; (II) all algebras with right units in which every element is a left-hand divisor of zero, and their isotopes; (III) all algebras anti-isomorphic to those of class II; (IV) all algebras in which every element is both a left-hand and right-hand divisor of zero. Isotopically simple algebras of order n over an arbitrary field are constructed as follows: of class I, for all $n \neq 2$; of classes II and III, for all $n > 3$. Isotopically simple algebras of order 2 are (isotopic to) quadratic fields. Certain types of Lie algebras (class IV) are shown to be isotopically simple. New simple algebras of all orders and new division algebras of orders 4, 8 and countable infinity are defined in terms of quasigroup algebras and generalized quasigroup algebras. The notion of generalized quasigroup algebra stems from the extension problem for quasigroups, of which a complete solution is given here for quasigroups of arbitrary order. (Received October 6, 1943.)

5. Claude Chevalley: *Some properties of ideals in rings of power series.*

The paper is concerned mainly with the behavior of prime ideals in power series rings under extension of the basic field. The theorem of A. Weil on the existence of the field of definition of a prime ideal in the ring of polynomials is generalized for prime ideals in rings of power series. (Received October 8, 1943.)

6. M. M. Day: *Arithmetic of ordered systems.*

Operations of ordered addition and ordered multiplication of (partially) ordered systems are defined to include as special cases the ordinal and cardinal sum and product and the ordinal exponentiation of G. Birkhoff (Duke Math. J. vol. 9 (1942) pp. 283–302). If $(T, \geqq)$ and $(S_t, \geqq)$ are ordered systems, the ordered product over $(T, \geqq)$ of the $(S_t, \geqq)$ has as elements the functions f defined on T with $f(t)$ in S_t for every t: $f \geqq' f_1$ means that if $f(t) \neq f_1(t)$ there exists $t_1 \geqq t$ such that $f(t_1) > f_1(t_1)$ (essentially lexicographic ordering); the relation $\geqq$ in the ordered product is the least transitive relation including $\geqq'$. The product over the system of integers of a family of two-element well-ordered systems is a simple example in which $\geqq'$ is not transitive. The principal results depend on the fundamental theorem: If $(T, \geqq)$ is a number and each $(S_t, \geqq)$ is transitive, the relation $\geqq'$ is almost transitive; that is if $f_1 \geqq' f_2 \geqq' f_3 \geqq' \cdots \geqq' f_n$, there exists f_0 such that $f_1 \geqq' f_0 \geqq' f_n$. (Received October 22, 1943.)

7. R. P. Dilworth: *A decomposition theorem for partially ordered sets.*

Let P be a partially ordered set. P has finite *width* k if k is the least integer such that every set of $k+1$ elements contains a comparable pair. It is shown that *if P has finite width k, then P is the set sum of k disjoint chains.* An almost trivial application yields the P. Hall theorem on representatives of sets. Indeed, the above

theorem seems to be the natural infinite extension of the P. Hall theorem. Application to distributive lattices gives the following result: *Let* $m=\max h(a)$ *where* $h(a)$ *is the number of elements covering an element* a *of a distributive lattice* L. *Then* L *is a sublattice of a direct product of* m *chains and* m *is the least number of chains for which this holds.* (Received October 22, 1943.)

8. Roy Dubisch and Sam Perlis: *On the radical of a non-associative algebra.* (Preliminary report.)

The radical of an associative algebra may be defined in many equivalent ways. In the case of a non-associative algebra most of these definitions provide subsets which are not always ideals or which lack many of the common properties of the radicals of associative algebras. One of these common properties is preserved by the definition made recently by Albert for non-associative algebras obeying a simple hypothesis, but most of the other properties are lost. It is the purpose of this paper to investigate several definitions yielding ideals with at least one of the desired, well known attributes of a radical, to see if any two of these ideals coincide or satisfy an inclusion relation and to see if any one of them has some properties of particular interest. The paper begins with the special case of alternative algebras by giving a new and simple proof of Zorn's theorem that the totality R of properly nilpotent elements forms an ideal. The method consists of showing that the definition given in 1941 by one of the authors provides an ideal H and that $H=R$. Albert's radical N is shown to coincide with H and R for this case. But there are examples of non-alternative algebras for which the property $H=R=N$ fails. (Received October 29, 1943.)

9. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in groups.*

Let π be a discrete group which acts as a group of left operators on an abelian group G. A function f of n variables in π and with values in G is called an n-cochain. The coboundary δf of f is an $(n+1)$-cochain defined by setting $\delta f(x_1, \cdots, x_{n+1}) = x_1[f(x_2, \cdots, x_{n+1})] - f(x_1x_2, \cdots, x_{n+1}) + f(x_1, x_2x_3, \cdots, x_{n+1}) \cdots \pm f(x_1, \cdots, x_{n-1}, x_nx_{n+1}) \mp f(x_1, \cdots, x_n)$. Since $\delta\delta=0$, the customary construction yields a cohomology group $H_n(\pi, G)$ of π with coefficients in G. By a suitable definition of a cup product, the groups $H_n(\pi, G)$ give rise to a cohomology ring. The group $H_1(\pi, G)$ is the group of crossed homomorphisms of π in G (functions such that $f(x_1x_2)=f(x_1)+x_1f(x_2)$) modulo the principal homomorphisms (functions $f(x)=x(g)-g$, for a constant $g\in G$). The group $H_2(\pi, G)$ is the group of group extensions of G by π, with the indicated operators. If π is represented as a factor group F/R of a free group F, then $H_n(\pi, G)=H_{n-2}(\pi, \text{Hom } (R, G))$, with the group π suitably operating on the group Hom (R, G) of homomorphisms of R into G. (Received November 5, 1943.)

10. Wade Ellis: *Relations satisfied by linear operators on a vector space.*

Let α, β be two linear operators on a linear vector space V of dimension n. If $n=2$, all general relations on α, β are consequences of the three known relations, of which two involve only one operator each (Cayley), and the third expresses $\alpha\beta+\beta\alpha$ in terms of $\alpha, \beta, 1$ (Laguerre). If $n>2$, there are, in addition to those of a similar type, other relations (with invariants of certain kinds as coefficients) which it is the purpose of this paper to find and express. When $n=3$, an important distinction is made between the case (I) (of simultaneous symmetrizability) where there is a coordinate system

in which α, β are both represented by symmetric matrices, and the case (II) where this is not true. In I the relations are expressed as determinants of order seven, with all their elements rational invariants. In II the coefficients must involve a quadratic irrationality δ; it is convenient to use determinants of order nine whose elements involve δ and rational invariants. For $\delta=0$, these relations reduce to those of case I. Special cases of the relations occur in connection with the study of certain differential equations of physics. (Received October 23, 1943.)

11. N. J. Fine and Ivan Niven: *The probability that a determinant be congruent to a* (mod m).

A complete solution is given to the problem of evaluating $P_n(a, m)$, the probability that a determinant of order n having integral elements be congruent to a modulo m. (Received October 15, 1943.)

12. A. L. Foster and B. A. Bernstein: *Symmetric definition and duality theorem for rings.*

It is shown that commutative rings with unit are capable of symmetric definition and possess an elementary principle of duality of which the Boolean algebra duality is an instance. In this paper only non-ideal theoretic consequences of this duality are treated, and applications are made to the special case of fields. (Received November 29, 1943.)

13. Ralph Hull: *A theorem on the unit groups of simple algebras.*

Let $\mathfrak{A}$ be a normal simple algebra of degree n over an algebraic number field $\mathfrak{F}$. If $\mathfrak{A}$ satisfies condition R, that is, $n>2$ or $\mathfrak{A}$ is unramified at at least one infinite prime place of $\mathfrak{F}$ when $n=2$, then any two distinct maximal orders of $\mathfrak{A}$ have distinct unit groups. The proof is based on the splitting-field theory of algebras $\mathfrak{A}$ which satisfy R (Eichler, Math. Zeit. vol. 43 (1938) pp. 481–494) and local arithmetical considerations. Condition R is indispensable in general. The theorem can be used to describe the Brandt groupoid of normal ideals of $\mathfrak{A}$. (Received October 18, 1943.)

14. Jakob Levitzki: *A characteristic minimal condition for semi-primary rings.*

A ring S is called semi-primary if the sum R of all two-sided nilpotent ideals of S is nilpotent and S/R is semi-simple. It has been proved by Hopkins (C. Hopkins, *Nil-rings with minimal condition for admissible left ideals*, Duke Math. J. vol. 4 (1938) pp. 664–667) that a ring S with minimal condition for left-ideals is semi-primary. But neither this condition, nor the weaker assumptions found later by other authors are necessarily satisfied by each semi-primary ring. In the present note it is shown that the following minimal condition is necessary as well as sufficient for semi-primary rings: Each descending chain of the form $L_1 \supset L_2 \supset L_3 \supset \cdots$, where the L_i are left-ideals containing R, and each descending chain of the form $A_1 \supset A_1 \cdot A_2 \supset A_1 \cdot A_2 \cdot A_3 \supset \cdots$, where the A_i are two-sided ideals contained in R, is finite. (Received October 27, 1943.)

15. G. Y. Rainich: *Noncommutative relations.*

The problem of factorization of polynomials in a ring discussed previously (abstract 48-1-51) leads to the consideration of systems of relations (in terms of multi-

plication and addition) on indeterminates (which first appear as indeterminate coefficients). The situation is studied from several points of view. I (Abstract point of view): One set of relations may imply another. A set may be contradictory. Examples are given of complete sets; a set is complete when any relation compatible with it is implied by it. II (Realization): Here a linear vector space is considered and linear operators on it which satisfy the same relations as those that are given. The connection with I is given by the fact that relations satisfied in an invariant subspace imply relations in the whole space. In III a ring is considered generated (with the aid of a field) by operators satisfying given relations. The special case when the relations involve multiplication only correspond to a group algebra. IV deals with relations satisfied by operators as a result of their being operators on a vector space of a given dimensionality. (Received October 23, 1943.)

16. H. E. Salzer: *New tables and facts involving sums of four tetrahedral numbers.*

The author has a second empirical theorem about tetrahedral numbers, that is, $(n^3-n)/6$ for integral n. Every tetrahedral number greater than 1 is the sum of four other non-negative tetrahedrals. This theorem has been verified for the first 200 cases in a table expressing every tetrahedral from 4 through 1373701 as a sum of four non-negative tetrahedrals. With the exception of 153, the first 200 triangular numbers $n(n+1)/2$ can each be expressed as the sum of four non-negative tetrahedrals. There are only 45 integers less than or equal to 1000 which cannot be expressed as the sum of four non-negative tetrahedrals. All numbers ending in 0, 5, or 6 which are less than or equal to 2006 are expressible as a sum of four non-negative tetrahedrals. This includes the first 201 cases of each type. It is interesting to note that the smallest example of a number ending in 4 which is not expressible as a sum of four non-negative tetrahedrals occurs at 1314. Thus here is an instance where a statement is true in the first 131 cases, but fails in the 132nd. (Received October 13, 1943.)

17. L. R. Wilcox: *Modularity in Birkhoff lattices.*

The following theorem connecting G. Birkhoff's upper semi-modular lattices with the author's M-symmetric lattices is proved. Let a lattice be called upper semi-modular if $a+b$ covers a, b when a and b cover ab; let a lattice be called M-symmetric if $(a+b)c=a+bc$ for every $a\leqq c$ implies $(d+c)b=d+cb$ for every $d\leqq b$; finally, let a lattice be called of finite dimensional type if every a, b with $a<b$ have a finite principal chain connecting them. Then a lattice of finite dimensional type is upper semi-modular if and only if it is M-symmetric. The purpose of this theorem is to replace the condition of Birkhoff, forceful only when some chain condition is assumed, by a strictly algebraic condition which is suitable for use in the infinite dimensional case. (Received October 19, 1943.)

Analysis

18. Stefan Bergman: *The determination of singularities of functions satisfying a partial differential equation from the coefficients of their series development.*

Let $U(z, \bar{z})=A_{00}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}z^m\bar{z}^n$ be a (complex) solution of the equation $L(U)\equiv U_{z\bar{z}}+a_1U_z+a_2U_{\bar{z}}+a_3U=0$ where a_k, $k=1, 2, 3$, are entire functions of two variables $z=x+iy$, $\bar{z}=x-iy$, x, y real. Using the results of the papers Rec. Math.

(Mat. Sbornik) N.S. vol. 2 p. 1169, and Trans. Amer. Math. Soc. vol. 53 (1943) p. 130, §7, the author shows that the positions (that is, the coordinates x, y) of singularities of $U(z, \bar{z})$ are determined by the sequence $\{A_{m0}\}$, $m=0, 1, 2, \cdots$, and are independent of the a_k, $k=1, 2, 3$. He investigates further the connections between other subsequences of $\{A_{mn}\}$ and the positions of singularities. For instance the A_{m1} and the $\alpha_{m0}^{(\nu)}$, $\nu=1, 2, 3$, $m=0, 1, 2, \cdots$, where $a_\nu=\sum \alpha_{mn}^{(\nu)} z^m \bar{z}^n$, determine the positions of the singularities. If $a_2=\bar{a}_1$ and a_3 is real and $u=\sum D_{mn} z^m \bar{z}^n$ is a real solution of $L(u)=0$, then the D_{m0} and the $\alpha_{m0}^{(1)}$, $m=0, 1, \cdots$, determine the positions of singularities. Finally the author shows that many properties of the singularities are determined by the A_{m0} and are independent of the a_k, $k=1, 2, 3$. (Received October 26, 1943.)

19. Lipman Bers and Abe Gelbart: *On a class of functions defined by certain partial differential equations.*

This paper contains a detailed mathematical treatment of the results presented in the Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188, and several new results on the properties of the functions satisfying the system of equations $\sigma_1(x)u_x=\tau_1(y)v_y$, $\sigma_2(x)v_y=-\tau_2(y)v_x$. (Received October 4, 1943.)

20. Lipman Bers and Abe Gelbart: *On functions satisfying certain systems of partial differential equations.*

This paper continues the study of the class of complex-valued functions $f(z)=u(x, y)+iv(x, y)$ where u and v satisfy the system of differential equations (1) $u_x=\tau_1(y)v_y$, $u_y=\tau_2(y)v_x$, τ_i being positive and analytic. With the aid of the function $E(\alpha, y)$ belonging to the class and corresponding to the analytic function $\exp(\alpha y)$ (cf. Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188), a "Laplace transformation" $f(z)=\int_\sigma E(-\alpha, z)g(\alpha)d\alpha$ is defined. Under suitable conditions $f(z)$ belongs to the class. If $g(z)\equiv 1$, $f(z)=z^{(-1)}(z)$ corresponds to the analytic function $1/z$. Its nth Σ-derivative corresponds to $1/z^n$, its Σ-integral to $\log z$. Properties of these functions are investigated. (Received October 4, 1943.)

21. D. G. Bourgin and C. W. Mendel: *Orthonormal sequences.*

Let $f(x)\in L_2$ be odd and periodic of period 2π. Suppose $\{f(nx)\}$, $n=1, 2, \cdots$, is an orthonormal sequence on $0\leq x\leq 2\pi$. The investigation of functions satisfying these conditions leads to certain natural subclasses. Explicit examples are given other than the trivial solutions $f(x)=\sin kx$. Properties of these solutions are developed and various subsidiary conditions are formulated under which the trivial solutions alone are possible. (Received October 22, 1943.)

22. Alfred Brauer: *On certain limits.*

The following theorem is proved: Let k and l be positive integers and $m=k+l-2$. Suppose that the functions $f(x)$ and $g(x)$ and their first $m+2$ derivatives exist and are bounded in the interval $0<x\leq h$. Suppose further that $f(x)$ is positive there. Finally suppose that $\lim f(x)=\lim f'(x)=\cdots=\lim f^{(k-1)}(x)=0$, but $\lim f^{(k)}(x)\neq 0$, and $\lim g(x)=\lim g'(x)=\cdots=\lim g^{(l-1)}(x)=0$, but $\lim g^{(l)}(x)\neq 0$. Then $\lim f(x)^{g(x)}=1$. In all limits $x\to+0$. (Received November 19, 1943.)

23. W. B. Caton and Einar Hille: *On the class of functions $H_p(1/2)$.*

One says that $f(z)\in H_p(1/2)$, $1<p\leq 2$, if $f(z)$ is holomorphic in $R(z)>1/2$ and

if $\int_{-\infty}^{\infty}|f(x+iy)|^p dy<M$ for all $x>1/2$. In this note a new proof is given of the well known theorem which states that $f(z)\in H_p(1/2)$ implies $f(z)=\int_0^\infty e^{-uz}F(u)du$, $F(u)e^{-u/2}\in L_q(0,\ \infty)$, $1/p+1/q=1$. The proof follows a general plan outlined in a paper of Hille (Compositio Math. vol. 6 (1938) pp. 93–102). (Received October 19, 1943.)

24. B. H. Colvin: *The expansion problem associated with a third order ordinary differential system of highly irregular type.*

The expansion problem considered is that associated with the third order ordinary differential system $y'''(x)+\lambda^3 y(x)=0$, $H_i(y)\equiv h_{i1}(\lambda)y''(a)+h_{i2}(\lambda)y'(a)+h_{i3}(\lambda)y(a)+h_{i4}(\lambda)y''(b)+h_{i5}(\lambda)y'(b)+h_{i6}(\lambda)y(b)=0$ $(i=1, 2, 3)$, in which the boundary conditions are of highly irregular type. The variable x is restricted to the real interval $a\leqq x\leqq b$: λ is an unbounded complex parameter and the coefficients $h_{ij}(\lambda)$ are polynomials in λ with complex coefficients. The general mode of attack is that recently employed by R. E. Langer in connection with highly irregular systems of the second order. By the use of some rather general notions of summability (first introduced by R. E. Langer) an expansion theory associated with such irregular systems is developed for integrable vectors $f(x)$ which fulfill certain conditions customarily imposed in such theories. The third order boundary systems considered include as special cases those discussed previously by J. W. Hopkins and L. E. Ward. The present theory, however, provides summable expansions for a class of functions far wider than that for which they obtained any results. (Received October 30, 1943.)

25. Paul Erdös and S. M. Ulam: *Some combinatorial problems in set theory.* Preliminary report.

The following problems seem of some interest: 1. Given a class K of subsets of a set E of power m, the class closed with respect to the operation of addition of less than m sets belonging to the class, and with respect to complementation, and with this additional property: For every subdivision of E into disjoint sets containing each more than one element, there exists a set in K which contains exactly one element of each of the sets of the subdivision. Must K under these conditions coincide with the class of all subsets of E? If $m=\aleph_0$, the answer is negative; otherwise, unsolved. 2. Let E be a set of power m. Is it possible to define $n<2^m$ completely additive measure functions on the class of all subsets of E so that each subset will be measurable in at least one measure function? 3. If the continuum hypothesis is true, every subset of the continuum can be obtained by Borel operations effected on sets that are additive subgroups of the real numbers. The statement that every subset can be obtained by Borel operations on sets consisting of rationally independent numbers is equivalent to the continuum hypothesis. (Received October 27, 1943.)

26. R. C. James: *Orthogonality and differentiability in normed linear spaces.*

Several definitions can be given of orthogonality in normed linear spaces. In this paper, it is said that an element x is orthogonal to y if and only if $\|x+ky\|\geq\|x\|$ for all k. Such orthogonality is neither symmetric ($x\perp y$ does not imply $y\perp x$) nor additive ($x\perp y$ and $x\perp z$do not imply $x\perp y+z$). However, this orthogonality is never vacuous, since for any x and y there exists an a such that $x\perp(ax+y)$. An equivalent definition of orthogonality is: "$x\perp y$ if $\lim_{n\to\infty}\|nx+y\|-\|nx\|=0$," provided $x\neq 0$, and it follows that this orthogonality is additive if and only if the norm is linearly Gateaux differentiable at each nonzero point. Furthermore, if this differential exists at a point x,

it is equal to $-a\|x\|$, where $x \perp (ax+y)$. This gives rise to a condition for the existence of a scalar product, and to a means of investigating the effect on a normed linear space of the existence of the differential of the norm. (Received October 29, 1943.)

27. Anne L. Lewis: *Sufficiency proofs for the problem of Bolza in the calculus of variations.*

This paper is mainly concerned with the establishment of sufficient conditions for a strong relative minimum for the general isoperimetric problem of Bolza in non-parametric form. The proof involves an expansion of the Hilbert integral and the use of a so-called quasi Mayer field, which is defined in terms of the variations of the slope functions and multipliers generally associated with a Mayer field. This procedure combines the field theory methods of Hestenes (Trans. Amer. Math. Soc. vol. 42 (1937) pp. 141–153; Duke Math. J. vol. 5 (1939) pp. 309–324) and the expansion methods of Reid (Trans. Amer. Math. Soc. vol. 42 (1937) pp. 183–190; Ann. of Math. (2) vol. 38 (1937) pp. 662–678) and utilizes certain dominant properties of the Weierstrass E-function. There are obtained, moreover, various results concerning the Weierstrass E-function which yield a Lindeberg theorem for the isoperimetric problem considered. (Received October 22, 1943.)

28. A. T. Lonseth: *The propagation of error in linear problems.*

Suppose that A is a bounded linear transformation in Hilbert space H which possesses a bounded inverse A^{-1}, and that C is in H. Then the solution $X=A^{-1}C$ of $AX=C$ is also in H, its norm $N(X)$ satisfying $N(X) \leq M(A^{-1})N(C)$, where $M(A^{-1})$ is the bound of A^{-1}. The error ξ induced in X by errors α in A and γ in C is considered; it is found that $N(\xi) \leq M(A^{-1})\{N(\gamma)+M(\alpha)N(X)\}/\{1-M(A^{-1})M(\alpha)\}$, provided that $M(A^{-1})M(\alpha)<1$. (This inequality generalizes readily to linear functional equations in a Banach space.) A case of special interest is that in which $A=I+K$, where I is the identity and $M(K)<1$; then (E. Hilb) A has a unique inverse A^{-1} and $M(A^{-1})<1/\{1-M(K)\}$. In this case bounds are also found for the components of ξ. The inequalities provide limits to the errors incurred in certain approximate methods for solving linear problems: infinite systems of linear equations in infinitely many unknowns (method of segments) and the related linear integral equations of first and second kinds. (Received October 20, 1943.)

29. Morris Marden: *A recurrence formula for the solutions of certain partial differential equations.*

Given the second order linear partial differential equation $L(U)=U_{zz}+AU_z+BU_{\bar{z}}+CU=0$, where A, B and C are analytic functions of the two complex variables $z=x+iy=re^{i\theta}$ and $\bar{z}=x-iy=re^{-i\theta}$. As proved by Bergman (Rec. Math. (Mat. Sbornik) N.S. vol. 44 (1937) pp. 1169–1198), there exists a function $E(z, \bar{z}, t)$ such that the operator $P(f)=\int_{-1}^{1}E(z, \bar{z}, t)f[(1/2)z(1-t^2)](1-t^2)^{-1/2}dt$, acting upon an arbitrary analytic function $f(z)$, produces a solution U of the equation $L(U)=0$. As the solutions so produced may be expanded in terms of the particular solutions $U_n=P(z^n)$, the study of the solutions U would be aided by the existence of recurrence relations among the solutions U_n. In this paper it is proved that when $\log E(z, \bar{z}, t)$ is an even or odd polynomial in t, there can be found $q+1$ functions $k_j(r, \theta, n)$ such that $\partial U_n/\partial r=k_0U_n+k_1U_{n+1}+\cdots+k_qU_{n+q}$, where q is the degree of $\log E$ as a polynomial in t if $\log E$ is an odd polynomial in t, but is only half this degree if $\log E$ is an even polynomial in t. The result is a generalization of a recurrence relation for Bessels' functions J_n. (Received October 7, 1943.)

30. K. L. Nielsen: *On operators for linear partial differential equations.*

Bergman (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155) has considered the relation between functions $U(z, \bar{z})$ satisfying certain partial differential equations $L(U) = U_{z\bar{z}} + aU_z + bU_{\bar{z}} + cU = 0$ and an associated analytic function $f(z)$ in terms of which the solution of $L(U) = 0$ can be expressed. He has introduced an operator $P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z[1-t^2]/2)(1-t^2)^{-1/2} dt$ which transforms the class of analytic functions of one complex variable into a certain class of functions $C(E)$ which represents solutions of $L(U) = 0$. The author and Ramsay (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 156–162) have found simple forms of $E = \exp\left[N(z, \bar{z})t^n + M(z, \bar{z})t^m\right]$ for given partial differential equations. In this paper the case of $E = \exp\left[\sum_{\nu=1}^{n} p_\nu(z)t^\nu\right]$ is developed and limiting theorems on m and n are found for the above expression of E. (Received October 22, 1943.)

31. W. T. Reid: *Expansion theorems for boundary problems of the calculus of variations.*

In a previous paper (Amer. J. Math. vol. 54 (1932) pp. 769–790) the author treated a two-point boundary problem associated with the general problem of Bolza in the calculus of variations, and established certain expansion theorems in terms of the characteristic solutions of the considered problem. The present paper is concerned with the proof of more refined expansion theorems for such a boundary problem. In particular, the general theorems herein proved provide improvements of the expansion theorems of Kamke (Math. Zeit. vol. 45 (1939) pp. 759–787; vol. 46 (1940) pp. 231–250 and pp. 251–286) for self-adjoint definite problems involving a single differential equation of even order. The paper contains results of the Riesz-Fischer type which include as very special cases the results of Krein (Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 923–933) for a self-adjoint problem involving a single differential equation of even order, and the more extensive results of this type for a self-adjoint problem involving a second order differential equation that have been established by Galbraith and Warschawski (Duke Math. J. vol. 6 (1940) pp. 318–340). (Received October 21, 1943.)

32. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* III.

When the method of solving functional equations outlined in previous reports in this Bulletin (abstracts 49-9-212, 213) is not applicable, the author notes that a procedure of restricting variables, or generalizing functions or variables, frequently leads to equations with solutions containing arbitrary functions. Perhaps the simplest example is offered by the associative equation $\phi\{x, \phi(y, z)\} = \phi\{\phi(x, y), z\}$. The equation $\phi\{x, \phi(y, x)\} = \phi\{\phi(x, y), x\}$ has the solution $\phi(x, y) = \alpha(x+y)$ and the equation $\phi\{x, f(y, z)\} = f\{\phi(x, y), z\}$ has the solution $\phi(x, y) = \alpha(x) \cdot y$, $f(x, y) = x \cdot \beta(y)$. Also the equation $\phi\{x_1, x_2, \cdots, x_n, \phi(y_1, y_2, \cdots, y_{n+1})\} = \phi\{\phi(x_1, x_2, \cdots, x_n, y_1), y_2, y_3, \cdots, y_{n+1}\}$ has the solution $\phi(x_1, x_2, \cdots, x_{n+1}) = x_1 \cdot \alpha(x_2, x_3, \cdots, x_n) \cdot x_{n+1}$ $(n > 1)$. The latter equation has the inverse $f(u, x_2, \cdots, x_n, v) = f(x_1, x_2, \cdots, x_{n+1})$ where $u = f(x_1, t_1, t_2, \cdots, t_n)$ and $v = f(x_{n+1}, t_1, t_2, \cdots, t_n)$. In the above the functions α and β are arbitrary. (Received October 21, 1943.)

33. Wolfgang Sternberg: *On difference equations.*

The author gives a solution of the often treated fundamental difference equation

$F(t+1)-F(t)=\phi(t)$. It is supposed that the given function $\phi(t)$ can be expanded in a Fourier series in every finite interval of t and, for simplicity, that $\phi(t)$ is continuous. The solution is $F(t)=-\phi(t)/2+\int_a^t\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_a^t\phi(\tau)\cos 2\pi k(t-\tau)d\tau$ where a is an arbitrary real constant. The above series is not a Fourier series in the usual sense, because the upper limit of the integrals is not constant, but the variable t itself. Some other solutions are given besides the above one. Finally the general linear difference equation of order n with constant coefficients $a_nF(t+n)+\cdots+a_1F(t+1)+a_0F(t)=\phi(t)$ is reduced to n equations of order 1 by the following theorem: If the characteristic equation $P(u)=a_nu^n+\cdots+a_1u+a_0=0$ has n different roots α_i $(i=1, 2, \cdots, n)$ and if $F_i(t)$ is an arbitrary solution of $F_i(t+1)-\alpha_iF_i(t)=\phi(t)/P'(\alpha_i)$, then $F_1(t)+F_2(t)+\cdots+F_n(t)$ is a solution of the above equation of order n. The case of multiple zeros of $P(u)$ can be dealt with in a similar way. (Received October 30, 1943.)

34. W. J. Thron: *Sets of convergence for continued fractions.* Preliminary report.

Let A be a set of complex numbers. Corresponding to every set A, a set $Z(A)$ is defined as follows: $1\in Z$; for every positive integer k, $1+a_1/1+a_2/1+\cdots+a_k/1\in Z$ if $a_i\in A$ $(i=1, 2, \cdots, k)$; the set Z contains no other elements. The following theorem has been established: for every set A the set $Z(A)$ either contains the point $z=0$ or has no point in common with the set $1-Z(A)$ ($v\in 1-Z$ if $1-v\in Z$). If $0\in Z(A)$ there exists a continued fraction $1+K(a_n/1)$, all of whose elements a_n are in A, which diverges by oscillation. Hence such a set A cannot be a set of convergence, even if convergence is understood to mean convergence in the wider sense. Thus a necessary condition for a set A to be a set of convergence for continued fractions $1+K(a_n/1)$ is that the sets $Z(A)$ and $1-Z(A)$ have no point in common. (Received November 17, 1943.)

35. W. J. Trjitzinsky: *Problems of representation and uniqueness for functions of a complex variable.*

The leading idea in the present extensive work (as in the author's earlier related memoirs in Ann. École Norm. (3) vol. 55 (1938) Fasc. 2 pp. 119–191, and in Acta Math. vol. 70 (1939) pp. 63–163) is that functions are studied which are not necessarily analytic but which at the same time are sufficiently specialized so as to come within the scope of classical analytic tools (like Lebesgue-Stieltjes integration). It appears that conditions of monogeneity in one sense or another (in a rather general sense) over sets in the complex plane, which may be without interior points, are the conditions which give the desired degree of specialization. On the other hand, the classes of functions so obtained are of such great vastness and the possibility of classifying these functions according to various uniqueness properties and subsequently studying them is so wide that there appears to be on hand a very extensive field for new investigation. (Received October 23, 1943.)

36. S. M. Ulam: *Theory of the operation of product of sets.* III. Preliminary Report.

Two subsets A, B of a product set E^n are called product isomorphic (abstract 47-9-406), if there exists a one-to-one transformation $T(x)$ of E into itself such that the transformation T^n: $x_1\to T(x_1), \cdots, x_n\to T(x_n)$ maps A into B: $T^n(A)=B$. Starting with a class K of subsets of E one obtains the projective class containing it by

considering in the product set E^n sets of the form $Z_1 \times Z_i \times Z_n$, where Z_i belongs to K, the Borel field over such sets and using the operation of projection and complementation a finite number of times. Necessary and sufficient conditions are found in terms of a special mapping of E into E^n preserving a given class of sets in E^n, for the product isomorphism in case of sets A, B in the projective class. The classical theories of Borel sets and of projective sets correspond to the case where K is the sequence of rational intervals. Many results of these theories hold for the case of a general K and thus show a purely combinatorial and not topological origin. This permits the formulation of an analogous theory in the even more general case of projective algebras (abstract 49-5-151). (Received October 22, 1943.)

37. S. E. Warschawski: *On Theodorsen's method of conformal mapping of nearly circular regions.*

The paper deals with the problem of determining the mapping function of a circle onto a nearly circular region. This problem has some practical importance in the theory of airfoils. Let C be a nearly circular closed Jordan curve: $\rho = \rho(\phi)$, $0 \leq \phi \leq 2\pi$, $(\rho, \phi$ polar coordinates) where $(1/\rho)(d\rho/d\phi)$ is continuous and $1 \leq \rho(\phi) \leq 1+\epsilon$ for some ϵ, $0 < \epsilon < 1$. Suppose that $w = f(z)$ $(f(0) = 0, f'(0) > 0)$ maps the circle $|z| = |re^{i\theta}| < 1$ conformally onto the interior of C. If $\arg f(e^{i\theta}) = \phi(\theta)$, the $\log (f(z)/z)$ for $z = e^{i\theta}$ may be written as $\log \rho[\phi(\theta)] + i(\phi(\theta) - \theta)$. Hence by Fatou's formula, $\phi(\theta) - \theta = (1/2\pi)\int_0^{\pi} \{\log \rho[\phi(\theta+t)] - \log \rho[\phi(\theta-t)]\} \cot (t/2)dt = H[\phi(\theta)]$, and the function $\phi(\theta)$ may thus be determined by solving this integral equation. Theodorsen (National Advisory Committee for Aeronautics, Report 411, 1932) and Theodorsen and Garrick (National Advisory Committee for Aeronautics, Report 452, 1933) developed a practical method for computing a solution of this equation by successive approximations. In the present paper the theoretical basis of this method is studied. Sufficient conditions for the curve C are established under which the approximations $\phi_0(\theta) = \theta$, $\phi_n(\theta) = H[\phi_{n-1}(\theta)]$ and their derivatives $\phi_n'(\theta)$ converge to $\phi(\theta)$ and $\phi'(\theta)$ respectively, and the errors $|\phi_n(\theta) - \phi(\theta)|$ and $|\phi_n'(\theta) - \phi'(\theta)|$ are estimated. (Received October 27, 1943.)

Applied Mathematics

38. L. W. Cohen and S. M. Ulam: *On the algebra of systems of vectors and some problems in kinematics.*

The properties of equivalence for systems of vectors as postulated for the mechanics of rigid bodies are studied in linear spaces. It is proved that any finite system of vectors in n-space is equivalent to a unique system of vectors collinear with the edges of an n-simplex. It is also proved that any such system is equivalent to a system of $[n/2]+1$ vectors. Similar theorems hold for infinitely many vectors and for spaces of infinitely many dimensions. The problem of topological invariants of trajectories of systems of n points with respect to arbitrary motions of the coordinate system is formulated and results are obtained for the case of three points. (Received October 22, 1943.)

39. A. H. Copeland: *The nature of turbulence.*

There are exhibited in this paper a number of flows which consist of series of disturbances distributed temporally but unfortunately not specially at random, and which satisfy the hydrodynamic equations together with appropriate boundary con-

ditions. In conformance with experimental evidence each disturbance becomes dissipated as time increases. The following motions are obtained: flow without obstructions with constant velocity at infinity, flow through a pipe, flow outside a rotating cylinder, flow past a wall, and flow between parallel walls. These solutions may be regarded as mathematical models displaying some but not all of the characteristics of physical turbulent motion. (Received October 30, 1943.)

40. Isaac Opatowski: *Isoperimetric problems in bending of cantilevers.*

The paper considers cantilevers characterized by the fact that any two cross sections are obtainable from each other by a transformation of dilatation. The surface bounding these cantilevers is representable in the parametric form: $y=F(x)f(t)$, $z=G(x)g(t)$; the beam's axis is taken as the x-axis. The load consists of the cantilever's own weight W and a concentrated force P at the free end. The problem of finding cantilevers which have an assigned type of cross section (that is, assigned expressions of f and g), assigned length and weight, and which are of uniform strength (S. Timoshenko, *Strength of materials*, 2nd ed., part 1, p. 209) and have a minimum deflection Y is of an isoperimetric type with an additional condition for F and G in the form of a Volterra integral equation. If W is neglected with respect to P the problem has no meaning because Y becomes proportional to W. However if, besides this, the condition of uniform strength is abandoned, the classical type of a simple isoperimetric problem is obtained. This is a generalization of a result given by Blasius for circular cross section (Zeitschrift für Mathematik und Physik vol. 62 (1914) pp. 182–197). (Received October 21, 1943.)

41. H. E. Salzer: *Supplementary calculation of coefficients for numerical integration with central differences.*

The coefficients M_{2s}, which are the $(2s)$th Bernoulli polynomials of order $2s$ for argument equal to s, divided by $(2s)!$, were calculated from M_{22} to M_{40} to 20 significant figures, from M_{42} to M_{52} to 8 significant figures, and finally from M_{54} to M_{62} to 20 decimals. (This extends the earlier calculation which went up to M_{20}, mentioned in an abstract of this Bulletin, 49-9-224.) With these coefficients it is possible to integrate functions that are tabulated at very wide intervals, even when the size of the interval is such that the differences do not fall off. Functions tabulated to an extremely large number of decimal places can be integrated to the same or greater accuracy, since we can use many differences. The ratio of consecutive coefficients, about 1/4, practically annuls the error that accumulates in successive differences when the tabulated function has a certain accuracy. This is an advantage over the Laplace formula. These coefficients are very useful for checking by quadrature the key values of functions whose integrals are known and for computing key values of functions defined by integrals. (Received October 29, 1943.)

42. L. R. Wilcox: *Theory of traffic light networks.* I.

Let points P_{ij} with coordinates (x_i, y_j) $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$ be given, and let a matrix function $A(t)=(a_{ij}(t))$ be given, where each $a_{ij}(t)$ is a step function with values 0, 1. The combined system $S=(P_{ij}, A(t))$ is called a rectangular system of traffic lights. Motivation of the definition lies in the obvious connection with ordinary systems of traffic lights, where $a_{ij}(t)=1$ (0) means that the light of P_{ij} at the time

t shows green (red) for the horizontal line through P_{ij} and red (green) for the vertical. Various questions are asked of S, notably those concerning the possibility of non-stop travel along all horizontal and vertical paths through the point network. Those systems S admitting "through" speeds are determined and for each such S all through speeds are found. Such problems are solved by the theory of simultaneous congruences. The results are applied to the determination of the synchronization for the "best" flow of traffic. Consideration of "almost" through speeds and more general problems in traffic control is deferred to a later paper. (Received October 19, 1943.)

43. František Wolf: *On a problem of L. M. K. Boelter. Viscous fluid forced through a heated vertical pipe.*

The temperature at the wall is prescribed to be $t^\circ = \lambda \coth ms + \mu$, independent of θ. The density and viscosity are analytic functions of t. There are four differential equations of which three are of the second order. Solving Stokes' hydrodynamical equations for a cylinder, we obtain a set of integro-differential equations which can be used for successive approximations. The chief difficulty is in determining the boundary value of the pressure which, generally, is determined by an integral equation. After some preliminary changes of the differential system, the Laplace transformation can be applied. This leads to an algebraic equation for the transform of the boundary value of the pressure. A later paper will be devoted to the study of the convergence of the successive approximations and to the problem of the existence of the solution. (Received October 28, 1943.)

Ergodic Theory

44. P. R. Halmos: *In general a measure preserving transformation is mixing.*

The first proof is given of the old standing conjecture announced in the title. "In general" means of course that the exceptional set is of the first category in one of the usual natural topologies (the strong neighborhood topology) for measure preserving transformations. The principal new and quite surprising fact used in the proof is that for any almost nowhere periodic transformation T the set of all conjugates of T, that is, the set of all STS^{-1}, is everywhere dense. (Received October 26, 1943.)

Geometry

45. V. G. Grove: *The transformation of Čech.*

The purpose of this paper is to give a simple geometric construction of the general transformation of Čech. This is accomplished by first constructing a two parameter family of quadrics having second order contact with a surface and associated in a simple manner with a conjugate net on the surface. The polar plane of a point on the tangent to a curve of the net with respect to a quadric of the family is related to that point by a general transformation of Čech. Geometrical characterizations are made for several particular transformations of Čech. (Received November 19, 1943.)

46. Janet MacDonald: *Conjugate nets in asymptotic parameters.*

This paper presents some contributions to the projective differential geometry of conjugate nets in asymptotic parameters on an analytic nonruled surface in ordinary space. The equation of the bundle of quadrics each of which has contact of at least

the third order with both curves of a conjugate net at a point is deduced, and the cones in the bundle and the intersections of the bundle with the tangent plane are studied. Certain polar relations with respect to the quadrics of the bundle are presented. Certain loci and envelopes are investigated at a point of the surface in connection with a pencil of conjugate nets. Among these are the principal cubic, which is the locus of the principal points, and the principal conic, which is the envelope of the principal joins. Davis (*Contributions to the theory of conjugate nets*, doctoral dissertation, Chicago, 1932) defined and studied several canonical configurations, considering the conjugate net as parametric. In the present paper a study is made of Davis's canonical configurations in asymptotic parameters. (Received October 21, 1943.)

47. E. J. Purcell: *Variety congruences of order one in n-dimensional space.*

A variety congruence of order one in $[n]$ is an algebraic ∞^k-system of varieties, each of dimension $n-k$ and order h, in n-dimensional projective space, such that through a generic point of $[n]$ one and only one V^h_{n-k} of the system passes (k any positive integer not greater than n, and h any positive integer). The results of very many writers on Cremona transformations, Cremona involutions, $(1, m)$ correspondences, and line or curve congruences of order one can be obtained by specializing this paper. (Received October 7, 1943.)

Topology

48. K. T. Millsaps: *A note on generalized Hilbert space.*

By analyzing an example formulated by A. Tychonoff, Math. Ann. vol. 111 (1935) p. 768, the spaces H^ρ, $0 \leqq \rho \leqq \infty$, are defined in a manner analogous to that for classical Hilbert space; some basic properties such as linearity, necessary and sufficient conditions for normability, completeness, separability, and sufficient conditions for local non-convexness are proved. (Received October 30, 1943.)

49. R. L. Moore: *Concerning webs in the plane.*

Among other things it is shown that if a compact plane continuum contains a web it is one. (Received November 25, 1943.)

50. M. E. Shanks: *Monotone decompositions of continua.*

In this paper the author considers the lattice $\mathcal{D}_m(X)$ of all monotone upper semi-continuous decompositions of the compactum X. This lattice is well suited for the study of the structure of continua. Two continua X and Y are homeomorphic if and only if there is an isomorphism carrying $\mathcal{D}_m(X)$ onto $\mathcal{D}_m(Y)$ which makes simple decompositions correspond to simple decompositions. By means of a factorization theorem it is shown that if X is a dendrite or a linear graph then $\mathcal{D}_m(X)$ is isomorphic to $\mathcal{D}_m(A)$, where A is an arc. Spaces for which $\mathcal{D}_m(X)$ is isomorphic to $\mathcal{D}_m(A)$ are hereditarily locally connected. A class of continua called generalized dendrites is defined and characterized as those continua for which $\mathcal{D}_m(X)$ is a sublattice of the lattice of all upper semi-continuous decompositions of X. Both dendrites and Knaster continua are generalized dendrites, and both make $\mathcal{D}_m(X)$ distributive. A characterization of dendrites is obtained. (Received October 22, 1943.)

51. G. E. Schweigert: *Periodic analysis for onto-homeomorphisms of semi-locally-connected continua.* Preliminary Report.

$T(S)=S$ denotes any homeomorphism of a semi-locally-connected continuum. The least (invariant) A-set A which contains all cyclic elements of finite period is non-vacuous. Moreover $T(A)=A$ is componentwise periodic, that is, components of the complements of invariant A-sets in A have finite periods. It is shown that the action of $T(A)=A$ resembles that of an elementwise periodic homeomorphism (each cyclic element has finite period) provided certain free cyclic chains are used instead of cyclic elements. This involves a classification of the orbits of cyclic elements E with infinite period by means of the integer n for which the cyclic chain $C(E, T^n(E))$ has no invariant cyclic element (a property first studied by Ayres). It is also shown that the orbits of the components of $S-A$ are influenced by the periodicity in A and hence a pattern is formed for the action of $T(S)=S$ in the large. (Received October 21, 1943.)

52. G. T. Whyburn: *Mapping classes for locally connected continua.*

If A and B are locally connected continua, conditions are developed under which the limit mapping of a uniformly convergent sequence of continuous transformations of A onto B will be monotone, interior or quasi-monotone respectively. In particular it is shown that the limit mapping of any uniformly convergent sequence of quasi-monotone mappings of A onto B is itself quasi-monotone. Thus the class of all quasi-monotone mappings of A onto B is closed in the space B^A of all mappings of A onto B. It results from this that the limit of a uniformly convergent sequence of interior mappings of A onto B will be interior provided it is light. (Received October 22, 1943.)

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